**Math 322** **Midterm I**

**Name:** ______________________________

## Notes:

1. Show all of your work, present it neatly, and explain what you are doing. In particular, write complete sentences, either using words or mathematical symbols.
2. **You will only receive credit for the work that is shown.**
3. You can score a maximum of 130 points on this test. So **there are 10 "bonus" points**.
4. It is **more important** to leave out a problem and do all of the others in depth, than to do a little bit of each problem and finish none.

| Question | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | **Total** |
|---|---|---|---|---|---|---|---|---|---|---|
| Score | /20 | /20 | /20 | /25 | /5 | /10 | /10 | /10 | /10 | **/120** |

1. Consider the function $f(z) = \dfrac{i}{z-1}$.

a) [5 points] Write down the Cauchy-Riemann equations for an analytic function $u(x,y) + i\,v(x,y)$.

$\boxed{u_x = v_y \quad \& \quad u_y = -v_x}$ are the Cauchy-Riemann equations for $u(x,y) + i\,v(x,y) = f(z)$.

b) [12 points] Use the Cauchy-Riemann equations to show that the function $f$ above is analytic everywhere, except at $z = 1$.

$$u(x,y) + i\,v(x,y) = f(z) = \frac{i}{z-1} = \frac{i}{(x-1)+iy} = \frac{i[(x-1)-iy]}{(x-1)^2+y^2}$$

$$= \frac{y}{(x-1)^2+y^2} + i\,\frac{x-1}{(x-1)^2+y^2}$$

So $u(x,y) = \dfrac{y}{(x-1)^2+y^2}$ and $v(x,y) = \dfrac{x-1}{(x-1)^2+y^2}$.

Now, $u_x = \dfrac{-2(x-1)y}{((x-1)^2+y^2)^2}$ and $v_y = \dfrac{-2(x-1)y}{((x-1)^2+y^2)^2}$

So $u_x = v_y$.

Moreover, $u_y = \dfrac{(x-1)^2+y^2-2y^2}{[(x-1)^2+y^2]^2} = \dfrac{(x-1)^2-y^2}{((x-1)^2+y^2]^2}$

and $v_x = \dfrac{(x-1)^2+y^2-2(x-1)^2}{[(x-1)^2+y^2]^2} = \dfrac{-(x-1)^2+y^2}{[(x-1)^2+y^2]^2} = -u_y$

So $f(z)$ satisfies the Cauchy-Riemann equations everywhere, except at $z=1$. It is analytic, except at $z=1$.

c) [3 points] Is the function $f$ entire? Why or why not?

The function is <u>not entire</u> since it is not analytic at $z=1$.

2. Consider the function $u(x,y) = x\,(x^2 - 3\,y^2)$.

a) [5 points] Show that $u$ is harmonic.

$u_x = 3x^2 - 3y^2 \quad u_{xx} = 6x \quad$ so $\quad u_{xx} + u_{yy} = 6x - 6x = 0$

$u_y = -6xy \quad u_{yy} = -6x$

Since $\Delta u = 0$, the function $u$ is harmonic.

b) [12 points] Find the harmonic conjugate of $u$, and call this function $v(x,y)$. Explain what you are doing.

$u(x,y) = x^3 - 3xy^2$. We look for a function $v$ such that $u + iv$ is analytic. With the Cauchy-Riemann equations, we have $u_x = v_y$ and $u_y = -v_x$.

$u_x = v_y \Rightarrow 3x^2 - 3y^2 = v_y$

$\Rightarrow v(x,y) = 3x^2y - y^3 + h(x)$

$\Rightarrow v_x = 6xy + h'(x)$

Since we need $v_x = -u_y$, we have $6xy + h'(x) = 6xy$

So $h'(x) = 0$ i.e. $h = c = $ constant.

We choose $c = 0$ and get $\boxed{v(x,y) = 3x^2y - y^3}$.

c) [3 points] Express $u(x,y) + i\,v(x,y)$ as a function of the variable $z = x + i\,y$. Show all of your work.

$u + iv = x^3 - 3xy^2 + i\,(3x^2y - y^3) = (x + iy)^3 = z^3.$

3. Evaluate the following expressions and write the result in the form $a + i\,b$, where $a$ and $b$ are real. Give **exact answers** (do not approximate them numerically). If a function is multi-valued, give **all of the possible values** of that function.

a) [3 points] $\mathrm{Ln}\left(\sqrt{3}+i\right) = \ln\left|\sqrt{3+1}\right| + i\arctan\left(\frac{1}{\sqrt{3}}\right)$

$$= \ln(2) + i\,\frac{\pi}{6}.$$

b) [3 points] $\ln\left(\sqrt{3}+i\right) = \ln(2) + i\,\frac{\pi}{6} + 2p\,i\,\pi \qquad p \in \mathbb{Z}.$

c) [3 points] $\cosh(2+i\pi) = \dfrac{e^{2+i\pi} + e^{-2-i\pi}}{2} = \dfrac{e^{2}(-1) + e^{-2}(-1)}{2}$

$$= -\cosh(2).$$

d) [3 points] $\exp(7i\pi+1) = e\left[\cos(7\pi) + i\sin(7\pi)\right]$

$$= e(-1+0) = -e$$

e) [8 points] $\sqrt[3]{-2+2i} = \sqrt{2}\,e^{-i\pi/4}$ or $\sqrt{2}\,e^{-i\frac{\pi}{4}+i\frac{2\pi}{3}}$

or $\sqrt{2}\,e^{-i\frac{\pi}{4}+i\frac{4\pi}{3}}$

Indeed, we need to solve $-2+2i = z^3$. Since $-2+2i = \sqrt{8}\,e^{-\frac{3i\pi}{4}}$, this reads $z^3 = \sqrt{8}\,e^{-3i\pi/4}$ i.e. $z = \sqrt{2}\,e^{-i\pi/4 + p\frac{2\pi i}{3}}$ $\quad p = 0, 1, 2.$

4. Consider the function $g(z) = \dfrac{1}{\bar{z}}$.

a) [4 points] Given that it involves a term in $\bar{z}$, do you expect $g(z)$ to be differentiable at $z = 1$? Why or why not?

We do not expect $g(z)$ to be differentiable since $\bar{z}$ is not analytic in $z$.

b) [4 points] Express the following difference quotient in terms of $\Delta x$, and $\Delta y$: $Q(\Delta x, \Delta y) = \dfrac{g(1+\Delta z) - g(1)}{\Delta z}$, where $\Delta z = \Delta x + i\,\Delta y$. **Do not try to separate the real part of $Q$ from its imaginary part.**

$$Q(\Delta x, \Delta y) = \frac{g(1+\Delta z) - g(1)}{\Delta z} = \frac{1}{\Delta z}\left[\frac{1}{\overline{1+\Delta z}} - \frac{1}{1}\right]$$

$$= \frac{1}{\Delta x + i\Delta y}\left[\frac{1}{1+\Delta x - i\Delta y} - 1\right] = \frac{1}{\Delta x + i\Delta y} \cdot \frac{-\Delta x + i\Delta y}{1+\Delta x - i\Delta y}$$

$$= \frac{-\Delta x + i\,\Delta y}{(\Delta x + i\Delta y)(1+\Delta x - i\Delta y)}.$$

c) [10 points] Calculate the limit of $Q(\Delta x, \Delta y)$ as $\Delta z \to 0$, along the line of equation $\Delta y = t\,\Delta x$. Show all your work.

With $\Delta y = t\,\Delta x$, we have

$$Q(\Delta x, t\,\Delta x) = \frac{-\Delta x + i t\,\Delta x}{(\Delta x + i t \Delta x)\,(1 + \Delta x - i t\,\Delta x)} = \frac{-1 + i t}{(1 + i t)\,(1 + \Delta x - i t \Delta x)}$$

Since $\Delta z = \Delta x + i\,\Delta y = \Delta x + i\,t\,\Delta x = (1 + i t)\,\Delta x$, the limit $\Delta z \to 0$ along the line $\Delta y = t\,\Delta x$ corresponds to the limit $\Delta x \to 0$ in the expression of $Q(\Delta x, t\Delta x)$.

As $\Delta x \to 0$, $Q(\Delta x, t\,\Delta x) \to \dfrac{-1 + i t}{1 + i t} = \dfrac{(-1 + i t)\,(1 - i t)}{1 + t^2}$

$$= \frac{-1 + 2 i t + t^2}{1 + t^2}$$

i.e. $\boxed{\lim_{\Delta x \to 0} Q(\Delta x, t\,\Delta x) = \dfrac{-1 + 2 i t + t^2}{1 + t^2}}$.

d) [4 points] Using the result of part c), show that the function $g(z)$ is not differentiable at $z = 1$. Justify your answer.

Since the above limit depends on $t$, i.e. depends on the path followed to approach $z = 1$, the quantity $Q(\Delta x, \Delta y)$ does not have a limit as $z \to 1$. Therefore, g is not differentiable at $z = 1$.

e) [3 points] Is the function g analytic at z = 1? Why or why not?

Since $g$ is not differentiable at $z=1$, it is not analytic at $z = 1$.

5. [5 points] Give an example of a multi-valued function of a complex variable. Explain why it is multi-valued.

For instance, $\arg(z)$ is multivalued. Given a particular $z$, we can write $\arg(z) = \operatorname{Arg}(z) + 2p\pi$, where $p \in \mathbb{Z}$ is arbitrary.

6. [10 points] Use De Moivre's formula to express $\cos(4\,x)$ in terms of $\cos(x)$ and $\sin(x)$.

DeMoivre's formula is $e^{in\theta} = \left(e^{i\theta}\right)^n$, i.e.

$\cos(n\theta) + i\sin(n\theta) = \left[\cos(\theta) + i\sin(\theta)\right]^n$. With $n = 4$, this becomes

$$\cos(4\theta) + i\sin(4\theta) = \left[\cos(\theta) + i\sin(\theta)\right]^4$$

$$= \left[\cos^2(\theta) + 2i\sin(\theta)\cos(\theta) - \sin^2(\theta)\right]^2$$

$$= \cos^4(\theta) - 4\sin^2(\theta)\cos^2(\theta) + \sin^4(\theta) + 4i\sin(\theta)\cos^3(\theta)$$
$$- 2\cos^2(\theta)\sin^2(\theta) - 4i\sin^3(\theta)\cos(\theta)$$

$$= \left[\cos^4(\theta) - 6\sin^2(\theta)\cos^2(\theta) + \sin^4(\theta)\right]$$
$$+ 4i\left[\sin(\theta)\cos^3(\theta) - \sin^3(\theta)\cos(\theta)\right] = \cos(4\theta) + i\sin(4$$

By taking the real part of each side of the above equation, we see that $\boxed{\cos(4\theta) = \cos^4(\theta) + \sin^4(\theta) - 6\cos^2(\theta)\sin^2(\theta)}$.

7. [10 points] Is the following set of vectors linearly independent? Why or why not? $\left\{ \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Since $\begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, the 3 vectors are <u>not linearly independent</u>.

8. [10 points] Find a basis of the column space of the matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 4 & 2 & 1 \\ 7 & 5 & 1 \end{bmatrix}$.

Note that the columns of $A$ are the same vectors as in Problem 7 above. Explain what you are doing.

$$\text{Span}\left\{ \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} = \left\{ c_1 \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ c_1, c_2, c_3 \in \mathbb{R} \right\}$$

Since $\begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}$, any linear combination of the 3 vectors is in fact a linear combination of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}$. In other words,

$$\text{Span}\left\{ \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} = \left\{ c_1 \left( 2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} \right) + c_2 \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ c_1, c_2, c_3 \in \mathbb{R} \right\}$$

$$= \left\{ (2c_1 + c_3) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (c_1 + c_2) \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}, \ c_1, c_2, c_3 \in \mathbb{R} \right\}$$

$$= \left\{ A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + B \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}, \ A, B \in \mathbb{R} \right\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} \right\}.$$

Since $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}$ are not proportional, they are linearly independent, and therefore form a basis of the column space of $A$.

9. Consider the vector $U_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

a) [5 points] Find two vectors, $U_1$ and $U_2$, whose entries are **all non-zero**, and which, together with the vector $U_3$ above, form a basis for $\mathbb{R}^3$.

$U_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $U_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ are such that $\{U_1, U_2, U_3\}$ is a linearly independent set.

Indeed, $C_1 U_1 + C_2 U_2 + C_3 U_3 = 0 \iff \begin{cases} C_1 + 2C_2 = 0 \\ 2C_1 - C_2 = 0 \\ C_1 + C_2 + C_3 = 0 \end{cases}$

$\iff \begin{cases} C_1 + 4C_1 = 0 \\ C_2 = 2C_1 \\ 3C_1 + C_3 = 0 \end{cases} \iff \begin{cases} C_1 = 0 \\ C_2 = 0 \\ C_3 = 0 \end{cases}$.

b) [5 points] Explain why the set $\{U_1, U_2, U_3\}$ that you have found is a basis of $\mathbb{R}^3$.

Since $\mathbb{R}^3$ is 3-dimensional, any basis of $\mathbb{R}^3$ has exactly 3 vectors. Since $\{U_1, U_2, U_3\}$ is a linearly independent set, the 3 vectors $U_1$, $U_2$ and $U_3$ must span $\mathbb{R}^3$. If they did not, we would be able to find a basis of $\mathbb{R}^3$ with more than 3 vectors, which contradicts the fact that every basis has exactly 3 vectors.

Since $U_1$, $U_2$ and $U_3$ are linearly independent and span $\mathbb{R}^3$, they form a basis of $\mathbb{R}^3$.