Math 322 **Midterm II**

Name:

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Notes:

1. Show all of your work, present it neatly, and explain what you are doing. In particular, write complete sentences, either using words or mathematical symbols.
2. **You will only receive credit for the work that is shown.**
3. You can score a maximum of 130 points on this test. So **there are 10 "bonus" points**.
4. It is **more important** to leave out one or two sub-questions and do all of the others in depth, than to do a little bit of each and finish none.

Question	1	2	3	4	5	**Total**
Score	/38	/15	/30	/13	/34	**/120**

1) Consider the matrix

$$A = \begin{bmatrix} 3 & 0 & -1 \\ -2 & 2 & 1 \\ 8 & 0 & -3 \end{bmatrix}.$$

a) [10 points] Find the eigenvalues of A. Show all your work and explain what you are doing.

b) [15 points] Find the eigenvectors of the matrix A. Show all your work and explain what you are doing.

c) [3 points] What are the algebraic and geometric multiplicities of each of the eigenvalues that you found in part a)? Explain.

d) [10 points] Show that the eigenvectors you found in part b) form a basis of $\mathbb{R}^3$. Show all your work and explain what you are doing.

2) Consider the system of differential equations

$$\begin{cases} \dfrac{dx_1}{dt} = 3x_1 - x_3 \\ \dfrac{dx_2}{dt} = -2x_1 + 2x_2 + x_3 \\ \dfrac{dx_3}{dt} = 8x_1 - 3x_3 \end{cases} \tag{1}$$

a) [5 points] Do you expect system (1) to have a unique solution near the initial condition $x_1(0) = 1$, $x_2(0) = 2$, $x_3(0) = 3$? Why or why not?

b) [10 points] Find the general solution to system (1). **You may use the results of question 1) above.** Explain what you are doing.

3) Consider the following functions

$$y_1(x) = e^{5x}, \qquad y_2(x) = xe^{5x}, \qquad y_3(x) = e^{-5x}.$$

a) [10 points] Are the functions y_1, y_2 and y_3 linearly independent? Why or why not?

b) [10 points] Find a differential equation such that the functions

$$y_1(x) = e^{5x}, \qquad y_2(x) = xe^{5x}, \qquad y_3(x) = e^{-5x}$$

form a basis of the set of solutions to that equation. Explain your reasoning.

c) [5 points] Check that $y_1(x) = e^{5x}$ is a solution of the differential equation you found in part b). Show all of your work.

d) [5 points] Find a solution to the differential equation you found in part b) that satisfies the following initial conditions:

$$y(0) = 1,\ y'(0) = 0, \text{ and } y''(0) = 50.$$

Explain what you are doing.

4) Consider the following system of equations

$$\begin{cases} x_1 + 3x_2 = a \\ 2x_1 + 6x_2 = b \end{cases} \tag{2}$$

where a and b are two real numbers.

a) [3 points] On what condition on a and b is system (2) consistent? Explain.

b) [2 points] Give an example of values of a and b for which system (2) is not consistent.

c) [2 points] Give an example of values of a and b, **both non zero**, for which system (2) is consistent.

d) [6 points] Find the general solution of system (2) with the values of a and b that you chose in part c). Explain what you are doing and show all of your work.

5) This question is concerned with applying the power series method to solve the differential equation

$$y'' - y' = 0. \qquad (3)$$

a) [10 points] Show that the coefficients a_n of the power series solution of (3)

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

satisfy the following recursion relation:

$$a_{n+2} = \frac{a_{n+1}}{n+2}, \qquad n = 0, 1, 2, \ldots$$

Show all your work and explain what you are doing.

b) [5 points] Use the above recursion relation to express the coefficients a_2, a_3, a_4, a_5 and a_6 in terms of a_1.

c) [5 points] Use the results of part b) to write the first 7 terms of the power series expansion of y. Your answer should be in terms of a_0 and a_1.

d) [14 points] Use the results of part c) to write the power series expansions of two linearly independent solutions to (3). Do you recognize what these solutions are? Does it make sense? Explain.