Math 322 **Midterm II**

Name:

Notes:

1. Show all of your work, present it neatly, and explain what you are doing. In particular, write complete sentences, either using words or mathematical symbols.
2. **You will only receive credit for the work that is shown.**
3. You can score a maximum of 130 points on this test. So **there are 10 "bonus" points**.
4. It is **more important** to leave out one or two sub-questions and do all of the others in depth, than to do a little bit of each and finish none.

Question	1	2	3	4	5	**Total**
Score	/38	/15	/30	/13	/34	**/120**

1) Consider the matrix

$$A = \begin{bmatrix} 3 & 0 & -1 \\ -2 & 2 & 1 \\ 8 & 0 & -3 \end{bmatrix}.$$

a) [10 points] Find the eigenvalues of A. Show all your work and explain what you are doing.

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 0 & -1 \\ -2 & 2-\lambda & 1 \\ 8 & 0 & -3-\lambda \end{vmatrix} = (2-\lambda) \begin{vmatrix} 3-\lambda & -1 \\ 8 & -3-\lambda \end{vmatrix}$$

$$= (2-\lambda)(\lambda^2 - 9 + 8) = (2-\lambda)(\lambda^2 - 1) = (2-\lambda)(\lambda - 1)(\lambda + 1)$$

So the eigenvalues of A are $\boxed{\lambda = 2,\ \lambda = 1 \text{ and } \lambda = -1}$.

b) [15 points] Find the eigenvectors of the matrix A. Show all your work and explain what you are doing.

$d=2$ We look for X such that $(A - d I) X = 0$.

$$A - 2I = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 0 & 1 \\ 8 & 0 & -5 \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$(A-2I)X = 0 \iff \begin{cases} x - z = 0 \\ -2x + z = 0 \\ 8x - 5z = 0 \end{cases} \iff x = z = 0.$$

So one eigenvector is $\boxed{U_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}$.

$d=1$ $$(A - I) X = 0 \iff \begin{bmatrix} 2 & 0 & -1 \\ -2 & 1 & 1 \\ 8 & 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\iff \begin{cases} 2x - z = 0 \\ -2x + y + z = 0 \\ 8x - 4z = 0 \end{cases} \iff \begin{cases} z = 2x \\ y = 0 \end{cases}$$

An eigenvector for $d=1$ is $\boxed{U_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}}$.

$d=-1$ $$(A + I) X = 0 \iff \begin{bmatrix} 4 & 0 & -1 \\ -2 & 3 & 1 \\ 8 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\iff \begin{cases} 4x - z = 0 \\ -2x + 3y + z = 0 \\ 8x - 2z = 0 \end{cases} \iff \begin{cases} z = 4x \\ 2x + 3y = 0 \end{cases}$$

An eigenvector for $d=-1$ is $\boxed{U_{-1} = \begin{bmatrix} -3 \\ 2 \\ -12 \end{bmatrix}}$.

c) [3 points] What are the algebraic and geometric multiplicities of each of the eigenvalues that you found in part a)? Explain.

All of the eigenvalues of A have algebraic multiplicity 1. Since the geometric multiplicity is never larger than the algebraic multiplicity, it is also equal to 1. So $M_\lambda = m_\lambda = 1$ for $\lambda = -1, 1, 2$.

d) [10 points] Show that the eigenvectors you found in part b) form a basis of $\mathbb{R}^3$. Show all your work and explain what you are doing.

We found 3 eigenvectors. They will form a basis of $\mathbb{R}^3$ (which is 3-dimensional), provided they are linearly independent.

To check for linear independence, we calculate the determinant of the matrix $[v_2 \ v_1 \ v_{-1}]$.

$$\left| v_2 \ v_1 \ v_{-1} \right| = \begin{vmatrix} 0 & 1 & -3 \\ 1 & 0 & 2 \\ 0 & 2 & -12 \end{vmatrix} = - \begin{vmatrix} 1 & -3 \\ 2 & -12 \end{vmatrix} = -(-12+6)$$

$$= 6 \neq 0$$

Since $|v_2 \, v_1 \, v_{-1}| \neq 0$, the 3 vectors are linearly independent. They therefore form a basis of $\mathbb{R}^3$.

2) Consider the system of differential equations

$$\begin{cases} \dfrac{dx_1}{dt} = 3x_1 - x_3 \\ \dfrac{dx_2}{dt} = -2x_1 + 2x_2 + x_3 \\ \dfrac{dx_3}{dt} = 8x_1 - 3x_3 \end{cases} \qquad (1)$$

a) [5 points] Do you expect system (1) to have a unique solution near the initial condition $x_1(0) = 1, x_2(0) = 2, x_3(0) = 3$? Why or why not?

This system can be written as

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{where } A = \begin{bmatrix} 3 & 0 & -1 \\ -2 & 2 & 1 \\ 8 & 0 & -3 \end{bmatrix}.$$

Since the entries of A are constant and therefore continuous everywhere, we expect the system to have a unique solution for any initial condition. (Use existence theorem for linear systems of differential equations).

b) [10 points] Find the general solution to system (1). **You may use the results of question 1) above.** Explain what you are doing.

The general solution of $\frac{dX}{dt} = AX$ is obtained by finding the eigenvalues and eigenvectors of A.

Each (eigenvalue, eigenvector) pair gives a solution of the form $X_i = U_i e^{\lambda_i t}$ where U_i is the eigenvector & λ_i the eigenvalue.

The general solution is a linear combination of 3 linearly independent solutions, i.e.

$$X = C_1 U_2 e^{2t} + C_2 U_1 e^{t} + C_3 U_{-1} e^{-t}, \quad \text{i.e.}$$

$$\boxed{X = C_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{2t} + C_2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} e^{t} + C_3 \begin{bmatrix} -3 \\ 2 \\ -12 \end{bmatrix} e^{-t},}$$

where C_1, C_2 & C_3 are 3 arbitrary real constants.

3) Consider the following functions

$$y_1(x) = e^{5x}, \qquad y_2(x) = xe^{5x}, \qquad y_3(x) = e^{-5x}.$$

a) [10 points] Are the functions y_1, y_2 and y_3 linearly independent? Why or why not?

To check for linear independence, we calculate the Wronskian of y_1, y_2 & y_3.

$$W(y_1, y_2, y_3) = \begin{vmatrix} e^{5x} & x e^{5x} & e^{-5x} \\ 5e^{5x} & (5x+1)e^{5x} & -5e^{-5x} \\ 25e^{5x} & (25x+10)e^{5x} & 25e^{-5x} \end{vmatrix}$$

$$= e^{5x} \begin{vmatrix} 1 & x & 1 \\ 5 & 5x+1 & -5 \\ 25 & 25x+10 & 25 \end{vmatrix} = e^{5x} \begin{vmatrix} 0 & x & 1 \\ 10 & 5x+1 & -5 \\ 0 & 25x+10 & 25 \end{vmatrix}$$

$$= e^{5x} (-10) \begin{vmatrix} x & 1 \\ 25x+10 & 25 \end{vmatrix} = -10 e^{5x} (25x - 25x - 10)$$

$$= 100 e^{5x} \neq 0$$

Since their Wronskian is non-zero, the 3 functions are <u>linearly independent.</u>

b) [10 points] Find a differential equation such that the functions

$y_1(x) = e^{5x}, \qquad y_2(x) = xe^{5x}, \qquad y_3(x) = e^{-5x}$

form a basis of the set of solutions to that equation. Explain your reasoning.

Since the solutions y_1, y_2 and y_3 are in the form of exponentials or polynomials times exponentials, we look for an equation that is linear with constant coefficients. The roots of the characteristic polynomial should be 5, 5 and −5. The characteristic polynomial therefore reads

$$(\lambda-5)^2(\lambda+5) = 0 \iff (\lambda^2 - 10\lambda + 25)(\lambda+5) = 0$$

$$\iff \lambda^3 - 10\lambda^2 + 25\lambda + 5\lambda^2 - 50\lambda + 125 = 0$$

$$\iff \lambda^3 - 5\lambda^2 - 25\lambda + 125 = 0.$$

The linear ODE whose characteristic polynomial is $\lambda^3 - 5\lambda^2 - 25\lambda + 125 = 0$ is

$$\boxed{y''' - 5y'' - 25y' + 125y = 0}.$$

c) [5 points] Check that $y_1(x) = e^{5x}$ is a solution of the differential equation you found in part b). Show all of your work.

$y_1 = e^{5x} \quad y_1' = 5e^{5x} \quad y_1'' = 25e^{5x} \quad y_1''' = 125e^{5x}$

Substitution into $y''' - 5y'' - 25y' + 125y = 0$

gives $125 - 5\cdot 25 - 25\cdot 5 + 125 = 0$

$\Leftrightarrow \quad 0 = 0.$

So $y_1 = e^{5x}$ is a solution of $y''' - 5y'' - 25y' + 125y = 0$.

d) [5 points] Find a solution to the differential equation you found in part b) that satisfies the following initial conditions:

$$y(0) = 1,\ y'(0) = 0, \text{ and } y''(0) = 50.$$

Explain what you are doing.

The general solution of $y''' - 5y'' - 25y' + 125y = 0$ is $y(x) = C_1 e^{5x} + C_2 x e^{5x} + C_3 e^{-5x}$. We impose the initial conditions to find C_1, C_2 and C_3.

$y'(x) = 5C_1 e^{5x} + C_2(5x+1)e^{5x} - 5C_3 e^{-5x}$

$y''(x) = 25C_1 e^{5x} + C_2(25x+10)e^{5x} + 25C_3 e^{-5x}$

The initial conditions read

$$\begin{cases} 1 = y(0) = C_1 + C_3 \\ 0 = y'(0) = 5C_1 + C_2 - 5C_3 \\ 50 = y''(0) = 25C_1 + 10C_2 + 25C_3 \end{cases} \Leftrightarrow \begin{cases} C_3 = 1 - C_1 \\ 5 = 10C_1 + C_2 \\ 25 = 10C_2 \end{cases}$$

$$\Leftrightarrow \begin{cases} C_3 = 1 - 1/4 = 3/4 \\ C_1 = 5/20 = 1/4 \\ C_2 = 5/2 \end{cases}$$

So $\boxed{y(x) = \frac{1}{4}e^{5x} + \frac{5}{2}xe^{5x} + \frac{3}{4}e^{-5x}}$.

4) Consider the following system of equations

$$\begin{cases} x_1 + 3x_2 = a \\ 2x_1 + 6x_2 = b \end{cases} \qquad (2)$$

where a and b are two real numbers.

a) [3 points] On what condition on a and b is system (2) consistent? Explain.

$x_1 + 3x_2 = a \Rightarrow 2x_1 + 6x_2 = 2a$

Since we also have $2x_1 + 6x_2 = b$, the system is consistent only if $\boxed{2a = b}$.

b) [2 points] Give an example of values of a and b for which system (2) is not consistent.

For instance, $\boxed{a = b = 1}$.

c) [2 points] Give an example of values of a and b, **both non zero**, for which system (2) is consistent.

For instance, $\boxed{a = 1 \text{ and } b = 2}$.

d) [6 points] Find the general solution of system (2) with the values of a and b that you chose in part c). Explain what you are doing and show all of your work.

$$\begin{cases} x_1 + 3x_2 = 1 \\ 2x_1 + 6x_2 = 2 \end{cases} \Leftrightarrow x_1 = 1 - 3x_2$$

Then $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 - 3x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

So $$\boxed{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{x_p = \text{particular solution}} + \underbrace{x_2 \begin{bmatrix} -3 \\ 1 \end{bmatrix}}_{x_h = \text{solution to homogeneous system.}}}$$

5) This question is concerned with applying the power series method to solve the differential equation

$$y'' - y' = 0. \qquad (3)$$

a) [10 points] Show that the coefficients a_n of the power series solution of (3)

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

satisfy the following recursion relation:

$$a_{n+2} = \frac{a_{n+1}}{n+2}, \qquad n = 0, 1, 2, \ldots$$

Show all your work and explain what you are doing.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \qquad y'(x) = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

$$= \sum_{n=0}^{\infty} a_{n+1} (n+1) x^n$$

$$y''(x) = \sum_{n=1}^{\infty} a_{n+1} (n+1) n x^{n-1} = \sum_{n=0}^{\infty} a_{n+2} (n+2)(n+1) x^n$$

So $y'' - y' = 0$ reads $\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=0} a_{n+1}(n+1)x^n = 0$

i.e. $\sum_{n=0}^{\infty} \left[a_{n+2}(n+2)(n+1) - a_{n+1}(n+1)\right] x^n = 0$

i.e. $a_{n+2}(n+2)(n+1) - a_{n+1}(n+1) = 0$ for $n = 0, 1, \ldots$

i.e. $\boxed{a_{n+2} = \frac{a_{n+1}}{n+2} \quad \text{for } n = 0, 1, 2, \ldots}$.

b) [5 points] Use the above recursion relation to express the coefficients a_2, a_3, a_4, a_5 and a_6 in terms of a_1.

$\underline{n=0}$ $\quad a_2 = \frac{a_1}{3}$ $\qquad$ $\underline{n=1}$ $\quad a_3 = \frac{a_2}{3} = \frac{a_1}{6}$

$\underline{n=2}$ $\quad a_4 = \frac{a_3}{4} = \frac{a_1}{24} = \frac{a_1}{4!}$ $\qquad$ $\underline{n=3}$ $\quad a_5 = \frac{a_4}{5} = \frac{a_1}{5!}$

$\underline{n=4}$ $\quad a_6 = \frac{a_5}{6} = \frac{a_1}{6!}$

c) [5 points] Use the results of part b) to write the first 7 terms of the power series expansion of y. Your answer should be in terms of a_0 and a_1.

We have $y(x) = a_0 + a_1 x + a_2 x^2 + \dots$

i.e. $y(x) = a_0 + a_1 x + \frac{a_1}{2} x^2 + \frac{a_1}{3!} x^3 + \frac{a_1}{4!} x^4 + \frac{a_1}{5!} x^5 + \frac{a_1}{6!} x^6 + \dots$

$$= a_0 + a_1 \left(x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \dots \right)$$

d) [14 points] Use the results of part c) to write the power series expansions of two linearly independent solutions to (3). Do you recognize what these solutions are? Does it make sense? Explain.

From above, we see that 2 linearly independent solutions of $y'' - y' = 0$ are

$y_1(x) = 1$ (set $a_0 = 1$ & $a_1 = 0$).

$y_2(x) = x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots = e^x - 1$ (set $a_0 = 0$ & $a_1 = 1$).

These solutions are linearly independent because they are not proportional to one another.

The characteristic polynomial of equation (3) is $d^2 - d = 0$, i.e. $d = 0$ and $d = 1$. The general solution of (3) is of the form $C_1 + C_2 e^x$.

We see that $y_1(x)$ corresponds to $C_1 = 0$ and $C_2 = 1$

$y_2(x)$ " $C_1 = -1$ " $C_2 = 1$