Problems for Quiz 14

Math 322. Spring, 2007.

1. Consider the initial value problem (IVP) defined by the partial differential equation (PDE)

$$u_t = u_{xx} - 2u_x + u, \qquad 0 < x < 1, \;\; t > 0 \tag{1}$$

with boundary conditions

$$u(0,t) = 0, \qquad u(1,t) = 0, \tag{2}$$

and initial condition

$$u(x,0) = f(x). \tag{3}$$

You will use the method of separation of variables to find the solution to this problem.

(a) Look for a solution of the PDE of the form $u(x,t) = F(x)G(t)$ and set up the corresponding eigenvalue problems (Hint: You should use the boundary conditions (2) to set up the eigenvalue problem for $F(x)$).

Sol. Substituting $u(x,t) = F(x)G(t)$ into (1) gives

$$F(x)\dot{G}(t) = (F''(x) - 2F'(x) + F(x))G(t)$$

where $' = \frac{d}{dx}$ and $\dot{} = \frac{d}{dt}$, so

$$\frac{F''(x) - 2F'(x) + F(x)}{F(x)} = \frac{\dot{G}(t)}{G(t)} = \lambda,$$

where λ is a constant. Therefore

$$F''(x) - 2F'(x) + F(x) = \lambda F(x), \tag{4}$$

$$\dot{G}(t) = \lambda G(t). \tag{5}$$

The boundary conditions (2) become:

$$F(0)G(t) = 0, \qquad F(1)G(t) = 0,$$

so we need:

$$F(0) = 0, \qquad F(1) = 0. \tag{6}$$

Equation (4) and boundary conditions (6) define the eigenvalue problem for $F(x)$.

(b) Consider the eigenvalue problem for $F(x)$ that you found in part a). Is it in Sturm-Liouville form? Can you transform it into Sturm-Liouville form? (Hint: Use problem 6 of problem set 5.7 in your text book).

Sol. The eigenvalue problem for $F(x)$ is given by

$$F''(x) - 2F'(x) + F(x) = \lambda F(x)$$

with boundary conditions

$$F(0) = 0, \qquad F(1) = 0.$$

Recall that a Sturm-Liouville problem consists of a second order differential equation:

$$(p(x)y')' + q(x)y = \lambda\sigma(x)y \tag{7}$$

together with boundary conditions of the form

$$C_1 y(a) + C_2 y'(a) = 0, \qquad C_3 y(b) + C_4 y'(b) = 0. \tag{8}$$

Since equation (4) does not have the form of equation (7), the problem for $F(x)$ is *NOT* in Sturm-Liouville form.
Following problem 6 of problem set 5.7 in the textbook, we multiply equation (4) on both sides by e^{-2x} to get:

$$e^{-2x}F''(x) - 2e^{-2x}F'(x) + e^{-2x}F(x) = \lambda e^{-2x}F(x)$$

that can be written as

$$(e^{-2x}F'(x))' + e^{-2x}F(x) = \lambda e^{-2x}F(x), \tag{9}$$

that has the form given by (7) with $p(x) = q(x) = \sigma(x) = e^{-2x}$.
The boundary conditions for $F(x)$ given by (6) have the form (8) where $a = 0, b = 1, C_1 = C_3 = 1, C_2 = C_4 = 0$.
So, by considering equation (9) and boundary conditions (6) we are able to write the eigenvalue problem for $F(x)$ as a Sturm-Liouville problem.

(c) Consider again the eigenvalue problem for $F(x)$. Find the eigenvalues λ_n and the corresponding eigenfunctions $F_n(x)$.

The characteristic polynomial associated to equation (4) is

$$p(\mu) = \mu^2 - 2\mu + (1 - \lambda)$$

with roots

$$\mu = \frac{2 \pm \sqrt{4 - 4(1 - \lambda)}}{2} = 1 \pm \sqrt{\lambda}.$$

Therefore, the general solution for $F(x)$ is

$$F(x) = Ae^{(1+\sqrt{\lambda})x} + Be^{(1-\sqrt{\lambda})x}$$

We need to find the values of λ for which the boundary conditions (6) are satisfied.
Enforcing $F(0) = 0$ gives $A + B = 0$, so we set $B = -A$ to get

$$F(x) = A(e^{(1+\sqrt{\lambda})x} - e^{(1-\sqrt{\lambda})x}). \tag{10}$$

Enforcing $F(1) = 0$ implies (we are interested in $A \neq 0$):

$$e^{(1+\sqrt{\lambda})} - e^{(1-\sqrt{\lambda})} = 0,$$

which can be rewritten as

$$e^{2\sqrt{\lambda}} = 1.$$

Writing $1 = e^{i2\pi n}$ we obtain

$$2\sqrt{\lambda} = i2\pi n$$

so $\sqrt{\lambda} = i\pi n$ and $\lambda = -(\pi n)^2$. Therefore the eigenvalues are $\lambda_n = -(\pi n)^2$ for $n = 1, 2, \ldots$ (0 is not an eigenvalue as we shall see below).
The eigenfunctions are obtained by substututing $\sqrt{\lambda} = i\pi n$ into (10). We get

$$\begin{aligned} F_n(x) &= Ae^x(e^{i\pi nx} - e^{-i\pi nx}) \\ &= 2iAe^x \sin(n\pi x). \end{aligned}$$

Recall that eigenfunctions are defined up to multiplication by a scalar factor. We can therefore simply take

$$F_n(x) = e^x \sin(n\pi x).$$

Notice that $F_0 = 0$ so $n = 0$ does not define a nonzero eigenfunction. Therefore the eigenvalues are $\lambda_n = -(\pi n)^2$ for $n = 1, 2, \ldots$ and the corresponding eigenfunctions are $F_n(x) = e^x \sin(n\pi x)$.

(d) Find the functions $G_n(t)$ corresponding to the eigenvalues λ_n that you found in part c) and write down explicit expressions for the solutions $u_n(x,t) = F_n(x)G_n(t)$ of the PDE.

Sol. The general solution to equation (5) is

$$G(t) = Ce^{\lambda t},$$

for a constant C. Considering the particular values $\lambda_n = -(\pi n)^2$, we find

$$G_n(t) = C_n e^{-(\pi n)^2 t}.$$

We therefore get the following family of functions as solutions of the PDE (1):

$$u_n(x,t) = C_n e^{-(\pi n)^2 t} e^x \sin(n\pi x). \tag{11}$$

(e) Verify that the functions $u_n(x,t)$ that you found in part d) are indeed solutions of the PDE (1).

Sol. Differentiating the expression for $u_n(x,t)$ given by equation (11) gives:

$$\begin{aligned}
\frac{\partial u_n}{\partial t} &= -(\pi n)^2 C_n e^{-(\pi n)^2 t} e^x \sin(n\pi x)\\
\frac{\partial u_n}{\partial x} &= C_n e^{-(\pi n)^2 t} e^x \sin(n\pi x) + n\pi C_n e^{-(\pi n)^2 t} e^x \cos(n\pi x)\\
\frac{\partial^2 u_n}{\partial x^2} &= C_n e^{-(\pi n)^2 t} e^x \sin(n\pi x) + 2n\pi C_n e^{-(\pi n)^2 t} e^x \cos(n\pi x) - (n\pi)^2 C_n e^{-(\pi n)^2 t} e^x \sin(n\pi x)
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{\partial^2 u_n}{\partial x^2} - 2\frac{\partial u_n}{\partial x} + u_n &= C_n e^{-(\pi n)^2 t} e^x \sin(n\pi x) + 2n\pi C_n e^{-(\pi n)^2 t} e^x \cos(n\pi x)\\
&\quad -(n\pi)^2 C_n e^{-(\pi n)^2 t} e^x \sin(n\pi x) - 2C_n e^{-(\pi n)^2 t} e^x \sin(n\pi x)\\
&\quad -2n\pi C_n e^{-(\pi n)^2 t} e^x \cos(n\pi x) + C_n e^{-(\pi n)^2 t} e^x \sin(n\pi x)\\
&= -(n\pi)^2 C_n e^{-(\pi n)^2 t} e^x \sin(n\pi x)\\
&= \frac{\partial u_n}{\partial t},
\end{aligned}$$

so $u_n(x,t)$ is indeed a solution of (1).

(f) The solution to the IVP is obtained by the principle of superposition: $u(x,t) = \sum_n A_n u_n(x,t)$ where the constant coefficients A_n are chosen to satisfy the initial condition (3). Using your answer to part b) and your knowlege on orthogonal expansions arising from Sturm-Liouville problems, write an explicit expression for the coefficients A_n (your formula should involve $f(x)$).

Sol. By the principle of superposition our solution will have the form (the constants C_n are absorbed into A_n):

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-(\pi n)^2 t} e^x \sin(n\pi x) \tag{12}$$

Setting $t = 0$ and enforcing the initial condition $u(x,0) = f(x)$ gives

$$f(x) = \sum_{n=1}^{\infty} A_n e^x \sin(n\pi x) \tag{13}$$

¿From part b) we know that the functions $F_n(x) = e^x \sin(n\pi x)$ are the eigenfunctions of the Sturm-Liouville problem defined by equation (9) and the boundary conditions $F(0) = F(1) = 0$. So the A_n's are the coefficients in the *eigenfunction expansion* of $f(x)$. There are two ways of determining the value of these coefficients (both of them come from the idea of orthogonal expansions):

i. For a general Sturm-Liouville problem

$$(p(x)y')' + q(x)y = \lambda\sigma(x)y,$$

with boundary conditions

$$C_1 y(a) + C_2 y'(a) = 0, \qquad C_3 y(b) + C_4 y'(b) = 0,$$

any pair of eigenfunctions y_m, y_n, corresponding to different eigenvalues, λ_m, λ_n, are *orthogonal* with respect to the weight function $\sigma(x)$, meaning that

$$\langle y_m, y_n\rangle = \int_a^b \sigma(x) y_m(x) y_n(x) dx = 0 \qquad \text{if} \qquad m \neq n.$$

Under reasonable and quite general assumptions on $f(x)$, one can obtain an *orthogonal expansion* of the form

$$f(x) = \sum_n A_n y_n(x).$$

As a consequence of orthogonality of the eigenfunctions, one has the following formula for the coefficients A_n (see formula (4) in section 5.8 of your textbook):

$$A_n = \frac{\langle f, y_n\rangle}{||y_n||^2} = \frac{\int_a^b \sigma(x) f(x) y_n(x) dx}{\int_a^b \sigma(x)(y_n(x))^2 dx}.$$

In our case, the eigenfunctions are $F_n(x) = e^x \sin(n\pi x)$, and from part (b) we know that $\sigma(x) = e^{-2x}$. Hence

$$A_n = \frac{\int_0^1 e^{-2x} f(x) e^x \sin(n\pi x) dx}{\int_0^1 e^{-2x}(e^x \sin(n\pi x))^2 dx} = \frac{\int_0^1 e^{-x} f(x)\sin(n\pi x) dx}{\int_0^1 \sin^2(n\pi x) dx}.$$

Using the identity $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos(2\theta)$, one computes $\int_0^1 \sin^2(n\pi x) dx = \frac{1}{2}$, so

$$A_n = 2\int_0^1 e^{-x} f(x)\sin(n\pi x) dx. \tag{14}$$

ii. One can also obtain this formula for the coefficients A_n by noticing that equation (13) can be written as

$$f(x) = e^x \sum_{n=1}^{\infty} A_n \sin(n\pi x).$$

Multiplying both sides of the equation by e^{-x} and calling $g(x) = e^{-x} f(x)$ we write

$$g(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x),$$

so the A_n's are the coefficients in the sine series expansion of $g(x)$. Therefore the well known formula:

$$A_n = \frac{2}{L}\int_0^L g(x)\sin(n\pi x) dx$$

holds. In our case, $L = 1$ and $g(x) = e^{-x} f(x)$ so

$$A_n = 2\int_0^1 e^{-x} f(x)\sin(n\pi x) dx$$

that is exactly (14).

(g) Write the solution to the problem if

$$f(x) = 2e^x \sin(3\pi x) - e^x \sin(7\pi x).$$

Sol. One can use the formula (14) to determine the value of A_n. However it is much easier to notice that the function $f(x) = 2e^x \sin(3\pi x) - e^x \sin(7\pi x)$ is already expressed as an orthogonal expansion. Equation (13) gives

$$2e^x \sin(3\pi x) - e^x \sin(7\pi x) = \sum_{n=1}^{\infty} A_n e^x \sin(n\pi x)$$

which implies $A_3 = 2, A_7 = -1$ and $A_n = 0$ for any value of n that is not 3 or 7. Hence, formula (12) gives

$$u(x,t) = 2e^{-9\pi^2 t} e^x \sin(3\pi x) - e^{-49\pi^2 t} e^x \sin(7\pi x).$$

2. Consider the boundary value problem (BVP) defined by Laplace's equation

$$u_{xx} + u_{yy} = 0 \qquad \text{on the square} \quad 0 < x, y < 2 \tag{15}$$

subject to the boundary conditions

$$u(0,y) = 0, \qquad u(x,2) = 0, \qquad u(2,y) = 0, \qquad u(x,0) = 100\sin(\pi x/2).$$

Solve the BVP using the method of separation of variables.

Sol. We begin by looking for a solution of the form $u(x,t) = F(x)G(y)$. Substitution into (15) and the usual arguments gives:

$$\frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)} = k$$

for some constant k. Therefore

$$\begin{aligned} F''(x) &= kF(x), \qquad (16)\\ G''(y) &= -kG(y). \qquad (17)\end{aligned}$$

The boundary conditions $u(0,y) = 0$ and $u(2,y) = 0$ become

$$0 = F(0)G(y) \qquad F(2)G(y).$$

We are interested in $G(y)$ that is not identically zero, so we must require

$$F(0) = 0, \qquad F(2) = 0. \tag{18}$$

Equation (16) and boundary conditions (18) define an eigenvalue problem for $F(x)$. The general solution to (16) is

$$F(x) = Ae^{\sqrt{k}x} + Be^{-\sqrt{k}x}.$$

Enforcing $F(0) = 0$ gives $A + B = 0$, so we set $B = -A$ to get

$$F(x) = A(e^{\sqrt{k}x} - e^{-\sqrt{k}x}). \tag{19}$$

Enforcing $F(2) = 0$ implies (we are interested in $A \neq 0$):

$$e^{2\sqrt{k}} - e^{-2\sqrt{k}} = 0$$

which can be rewritten as

$$e^{4\sqrt{k}} = 1.$$

Writing $1 = e^{i2\pi n}$ we obtain

$$4\sqrt{k} = i2\pi n$$

so $\sqrt{k} = \frac{i\pi n}{2}$ and $k = -(\frac{\pi n}{2})^2$. Therefore the eigenvalues are $k_n = -(\frac{\pi n}{2})^2$ for $n = 1, 2, \ldots$ (0 is not an eigenvalue as we shall see below).

The eigenfunctions are obtained by substuting $\sqrt{k} = i\frac{\pi n}{2}$ into (19). We get

$$\begin{aligned} F_n(x) &= A\left(e^{i(\frac{\pi n}{2})x} - e^{-i(\frac{\pi n}{2})x}\right)\\ &= 2iA\sin\left(\frac{n\pi x}{2}\right). \end{aligned}$$

Recall that eigenfunctions are defined up to multiplication by a scalar factor. We can therefore simply take

$$F_n(x) = \sin\left(\frac{n\pi x}{2}\right).$$

Notice that $F_0 = 0$ so $n = 0$ does not define a nonzero eigenfunction. Therefore the eigenvalues are $k_n = -\frac{(\pi n)^2}{2}$ for $n = 1, 2, \ldots$ and the corresponding eigenfunctions are $F_n(x) = \sin\left(\frac{n\pi x}{2}\right)$.

Substitution of $k_n = -\frac{(\pi n)^2}{2}$ into equation (17) gives

$$G_n''(y) = \frac{(\pi n)^2}{2} G_n(y),$$

whose general solution is

$$G_n(y) = A_n e^{\frac{n\pi y}{2}} + B_n e^{-\frac{n\pi y}{2}}. \tag{20}$$

Substitution of the boundary condition $u(x, 2) = 0$ into the expression $u_n(x, y) = F_n(x)G_n(y)$ gives

$$0 = F_n(x)G_n(2)$$

so we require $G_n(2) = 0$. Enforcing this condition in (20) gives

$$A_n e^{n\pi} + B_n e^{-n\pi} = 0. \tag{21}$$

This is one linear equation for the unknowns A_n, B_n. Its solution is of the form

$$\begin{pmatrix} A_n \\ B_n \end{pmatrix} = C_n \begin{pmatrix} e^{-n\pi} \\ -e^{n\pi} \end{pmatrix},$$

for a constant C_n. One can chose any nonzero value for C_n and carry on the analysis. Putting $C_n = \frac{1}{2}$ and substituting into (20) gives

$$\begin{aligned} G_n(y) &= \frac{1}{2}\left(e^{\frac{n\pi y}{2} - n\pi} - e^{-\frac{n\pi y}{2} + n\pi}\right) \\ &= \sinh\left(\frac{n\pi}{2}(y-2)\right). \end{aligned}$$

Therefore we find

$$u_n(x, y) = \sin\left(\frac{n\pi x}{2}\right) \sinh\left(\frac{n\pi}{2}(y-2)\right).$$

Setting $u(x, y) = \sum_n A_n u_n(x, y)$ gives

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{2}\right) \sinh\left(\frac{n\pi}{2}(y-2)\right).$$

We now enforce the boundary condition at $y = 0$. Substituting $y = 0$ in the above expression and enforcing $u(x, 0) = 100\sin(\pi x/2)$ gives:

$$100\sin\left(\frac{\pi x}{2}\right) = \sum_{n=1}^{\infty} -A_n \sin\left(\frac{n\pi x}{2}\right) \sinh(n\pi).$$

It follows that

$$A_1 = -\frac{100}{\sinh(\pi)}, \qquad A_n = 0 \quad \text{for} \quad n \neq 1.$$

Therefore the solution to the problem is

$$u(x, y) = -\frac{100}{\sinh(\pi)} \left[\sin\left(\frac{\pi x}{2}\right) \sinh\left(\frac{\pi}{2}(y-2)\right)\right].$$

3. Consider the initial value problem (IVP) defined by partial differential equation (PDE)

$$u_t = u_{xx} \qquad 0 \le x \le 2, \quad t \ge 0$$

subject to the boundary conditions

$$u(0,t) = 0, \qquad u(2,t) = 0$$

and the initial condition

$$u(x,0) = \begin{cases} x, & \text{for } 0 < x < 1; \\ 2-x, & \text{for } 1 < x < 2 \end{cases}$$

Solve the IVP using the method of separation of variables.

sol. We start with looking for solution in the form

$$u(x,t) = F(x)G(t)$$

Method of separation of variable gives

$$\frac{F''(x)}{F(x)} = \frac{G'(t)}{G(t)} = \lambda$$

i.e.

$$F''(x) = \lambda F(x) \tag{22}$$

$$G'(t) = \lambda G(t) \tag{23}$$

Equation (22) has solution

$$F(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}.$$

Enforcing $F(0) = 0$ gives $A + B = 0$, so we set $B = -A$ to get

$$F(x) = A(e^{\sqrt{\lambda}x} - e^{-\sqrt{\lambda}x}). \tag{24}$$

Enforcing $F(2) = 0$ implies (we are interested in $A \neq 0$):

$$e^{2\sqrt{\lambda}} - e^{-2\sqrt{\lambda}} = 0$$

which can be rewritten as

$$e^{4\sqrt{\lambda}} = 1.$$

Writing $1 = e^{i2\pi n}$ we obtain

$$4\sqrt{\lambda} = i2\pi n$$

so $\sqrt{\lambda} = \frac{i\pi n}{2}$ and $\lambda = -(\frac{\pi n}{2})^2$. Therefore the eigenvalues are $\lambda_n = -(\frac{\pi n}{2})^2$ for $n = 1, 2, \dots$.

The eigenfunctions are obtained by substuting $\sqrt{\lambda} = i\frac{\pi n}{2}$ into (19). We get

$$\begin{aligned} F_n(x) &= A\left(e^{i(\frac{\pi n}{2})x} - e^{-i(\frac{\pi n}{2})x}\right) \\ &= 2iA\sin\left(\frac{n\pi x}{2}\right). \end{aligned}$$

Recall that eigenfunctions are defined up to multiplication by a scalar factor. We can therefore simply take

$$F_n(x) = \sin\left(\frac{n\pi x}{2}\right).$$

Then equation (23) has solution

$$G_n(t) = e^{-(\frac{\pi n}{2})^2 t}.$$

So,

$$u_n(x,t) = F_n(x)G_n(t) = e^{-(\frac{\pi n}{2})^2 t}\sin\left(\frac{n\pi x}{2}\right)$$

And thus,

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-(\frac{\pi n}{2})^2 t} \sin\left(\frac{n\pi x}{2}\right).$$

To enforce initial condition, let $t = 0$, we have

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{2}\right).$$

For any initial condition $u(x,0) = f(x)$ given by the problem, we will have the formula to compute A_n as follows

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{2}\right)dx.$$

In this particular problem, $L = 2$, so

$$\begin{aligned}
A_n &= \frac{2}{2}\int_0^2 f(x)\sin\left(\frac{n\pi x}{2}\right)dx \\
&= \int_0^1 x\sin\left(\frac{n\pi x}{2}\right)dx + \int_1^2 (2-x)\sin\left(\frac{n\pi x}{2}\right)dx \\
\int_0^1 x\sin\left(\frac{n\pi x}{2}\right)dx &= \frac{-x\cos\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}\Big|_0^1 + \int_0^1 \frac{\cos\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}dx \\
&= \frac{-x\cos\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}\Big|_0^1 + \frac{\sin\left(\frac{n\pi x}{2}\right)}{(\frac{n\pi}{2})^2}\Big|_0^1 \\
&= \frac{-\cos\left(\frac{n\pi}{2}\right)}{\frac{n\pi}{2}} + \frac{\sin\left(\frac{n\pi}{2}\right)}{(\frac{n\pi}{2})^2} \\
\int_1^2 (2-x)\sin\left(\frac{n\pi x}{2}\right)dx &= \int_1^2 2\sin\left(\frac{n\pi x}{2}\right)dx - \int_1^2 x\sin\left(\frac{n\pi x}{2}\right)dx \\
&= \frac{-2\cos\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}\Big|_1^2 + \frac{x\cos\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}\Big|_1^2 - \int_1^2 \frac{\cos\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}dx \\
&= \frac{-2\cos\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}\Big|_1^2 + \frac{x\cos\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}\Big|_1^2 - \frac{\sin\left(\frac{n\pi x}{2}\right)}{(\frac{n\pi}{2})^2}\Big|_1^2 \\
&= \frac{-2\cos\left(\frac{2n\pi}{2}\right)}{\frac{n\pi}{2}} + \frac{2\cos\left(\frac{n\pi}{2}\right)}{\frac{n\pi}{2}} + \frac{2\cos\left(\frac{2n\pi}{2}\right)}{\frac{n\pi}{2}} - \frac{\cos\left(\frac{n\pi}{2}\right)}{\frac{n\pi}{2}} - \frac{\sin\left(\frac{2n\pi}{2}\right)}{(\frac{n\pi}{2})^2} + \frac{\sin\left(\frac{n\pi}{2}\right)}{(\frac{n\pi}{2})^2}
\end{aligned}$$

Thus,

$$A_n = \frac{\sin\left(\frac{n\pi}{2}\right)}{(\frac{n\pi}{2})^2} + \frac{\sin\left(\frac{n\pi}{2}\right)}{(\frac{n\pi}{2})^2} = \frac{2\sin\left(\frac{n\pi}{2}\right)}{(\frac{n\pi}{2})^2} = \frac{8\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2}$$