Math 425/525 (Fall 2011) - Calculus I Contents

Problem I. Assume the function $f:\ \mathbb{R} \longrightarrow \mathbb{R}$ converges to $l \in \mathbb{R}$ as $x \to x_0$, $x_0 \in \mathbb{R}$.

1. What is the $\epsilon - \delta$ definition of the above statement? Draw a picture that illustrates this definition.
2. What is the definition of the above statement in terms of sequences? Draw a picture that illustrates this definition.
3. Show that the two definitions are equivalent.

Problem II

1. What does it mean for the set $S \subset \mathbb{R}$ to be dense in $\mathbb{R}$?
2. Is $\mathbb{Q}$ dense in $\mathbb{R}$? Why or why not?
3. Show that if S is dense in $\mathbb{R}$, then $\forall x_0 \in \mathbb{R}$, one can find a sequence of points in S that converges to x_0.
4. Give an example of a set S which is not dense in $\mathbb{R}$.
5. Is $\mathbb{Q}$ closed in $\mathbb{R}$? Why or why not?

Problem III

1. How do you show that a function is uniformly continuous?
2. How do you show that a function is not uniformly continuous?
3. Give an example of a function which is continuous but not uniformly continuous on $\mathbb{R}$. Justify your answer.
4. Give an example of a non-constant function that is uniformly continuous on $\mathbb{R}$. Justify your answer.
5. Is $x \longmapsto \sin(x)$ uniformly continuous on $\mathbb{R}$? Why or why not?

Problem IV. Consider the function f that gives the temperature T at a point along the equator as a function of its longitude θ (in degrees, between 0 and 360).

1. Explain why it is reasonable to consider that f is continuous. We will also assume that it is not constant, and that it is differentiable on $[0, 360]$.
2. Draw a possible graph of f as a function of θ.
3. Explain why f must have at least one maximizer and one minimizer.
4. If θ_0 is such that $f'(\theta_0) \neq 0$, explain why there is another point along the equator where the temperature is the same as the temperature at θ_0. Is that point unique? Why or why not?

5. Assume f is increasing on the interval $[\theta_1, \theta_2]$. Explain why this is equivalent to saying that $f'(\theta) \geq 0, \ \forall \theta \in [\theta_1, \theta_2]$.

Problem V

1. Carefully write down the theorem giving the derivative of the inverse of a function f. Look at the proof given in the book and discussed in class.
2. Why is J a neighborhood of $y_0 = f(x_0)$?
3. What can go wrong if f is not strictly monotone? Give an example.
4. What can go wrong if f is not continuous? Give an example.
5. What can go wrong if $f'(x_0)$ vanishes?
6. Where is the composition of limits used in the proof?
7. Where is the quotient property of limits used in the proof?
8. Where is the definition of differentiability of f used in the proof?