Real Analysis of One Variable

MATH 425

Overview

The field of real numbers

1. Field axioms for $(\mathbb{R}, +, .)$

2. Positivity axioms for $\mathbb{R} \to$ induce a natural order on $\mathbb{R}$
   - $a > b \iff a - b \in \mathcal{P}$
   - $a^2 > 0, \ \forall a \in \mathbb{R}^*$
   - In particular, $1 = 1^2 > 0$

3. Completeness axiom $\to$ non-empty sets bounded above (resp. below) have a supremum (resp. infimum)
   - This is used to define $\sqrt{x}$ for $x > 0$
   - And to show that $\mathbb{R} \setminus \mathbb{Q}$ is not empty

4. The triangle inequality

Subsets of $\mathbb{R}$

1. $\mathbb{N}$ is the intersection of all inductive subsets of $\mathbb{R}$
    - The above is used in proofs by induction
    - As a consequence, one can define functions of the form $x \mapsto x^n$, $x \in \mathbb{R}$ and $n \in \mathbb{N}$, as well as polynomials
2. We defined the following subsets of $\mathbb{R}$:
   $\mathbb{Z}$ (integers), $\mathbb{Q}$ (rationals) and $\mathbb{R} \setminus \mathbb{Q}$ (irrationals)
3. We proved the Archimedean property
4. We showed that $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense in $\mathbb{R}$
5. This allowed us to define the Dirichlet function

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Sequences

1. Definitions of convergence and of the limit of a converging sequence
2. Properties of limits of sequences: comparison lemma, linearity, product, and quotient properties
3. Sequences and sets
   - Every convergent sequence is bounded
   - Density and sequential density - applications to $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$
   - Closed sets
   - A monotone sequence converges if and only if it is bounded
   - Every sequence has a monotone subsequence
   - Every bounded sequence has a convergent subsequence
   - Intervals of the form $[a, b]$ are sequentially compact
4. A sequence of numbers is convergent if and only if it is Cauchy

Continuity

1. $\epsilon - \delta$ definitions of continuity and uniform continuity
2. Properties of continuous functions
   - Sum, product, quotient, and composition of continuous functions
3. Continuity and sequential continuity
   - Sequential definition of uniform continuity
   - Equivalence between continuity and sequential continuity
4. A continuous function on a closed bounded interval is uniformly continuous
5. Extreme value theorem
6. Intermediate value theorem

Continuity (continued)

1. Monotonicity and continuity
   - A monotone function is continuous if and only if it maps intervals to intervals
2. Continuity of inverse functions
   - If $f : I \to \mathbb{R}$ is strictly monotone and I is an interval, then $f^{-1} : f(I) \to \mathbb{R}$ is continuous
   - We can now define $x \mapsto x^r$, $x \in \mathbb{R}^+$ and $r \in \mathbb{Q}$
3. Limits $(\epsilon - \delta)$
   - Properties of limits: limit of the sum, product, quotient, and composition of two functions
   - "Sequential" definition of the limit of a function

Differentiation

1. The derivative as a limit of a difference quotient
2. Differentiable function are continuous
3. Properties of derivatives (inherited from properties of limits)
   - Product, sum, and quotient rules
   - Chain rule
   - Derivative of inverse function
   - We have an expression for the derivative of $x \mapsto x^r$, $x \in \mathbb{R}$ and $r \in \mathbb{Q}$
4. The mean value theorem and its consequences
   - Identity criterion
   - Criterion for strict monotonicity
   - Maximizers and minimizers of a function
5. * The Cauchy mean value theorem

* Applications

1. Assume there exists $F : (0, \infty) \to \mathbb{R}$ such that $F'(x) = \frac{1}{x}, \ \forall x > 0$ and $F(1) = 0$
    - Logarithm and its properties
    - Exponential as the inverse function of the logarithm
    - Properties of the exponential

2. Assume there exists $f : \mathbb{R} \to \mathbb{R}$ such that $f''(x) + f(x) = 0, \ \forall x \in \mathbb{R}$, with $f(0) = 1$ and $f'(0) = 0$
    - Cosine function
    - Sine function
    - Trigonometric identities
    - Tangent function
    - Inverse trigonometric functions

Integration

1. Upper and lower Darboux sums and their properties
2. Upper and lower integrals and definition of integrability
3. The Archimedes-Riemann theorem
    - Condition for integrability in terms of Darboux sums for an Archimedean sequence of meshes
    - Integral as a limit of upper and lower Darboux sums
    - Application: integrability of monotone and piecewise constant functions
4. Properties of the Riemann integral
    - Additivity over intervals, monotonicity, linearity

Integration (continued)

1. Continuity and integrability
   - A continuous function on a closed bounded interval is integrable
   - The value of the integral does not depend on the value of the function at the end points of the interval of integration
2. First fundamental theorem of calculus: integrating derivatives
3. Second fundamental theorem of calculus: differentiating integrals
   - If f is bounded and integrable on $[a, b]$, then F such that $F(x) = \int_a^x f$ is continuous on $[a, b]$
   - We can now calculate antiderivatives
   - We have a chain rule formula for integrals

* Applications

1. With the second fundamental theorem, we can now define $\ln(x) = \int_1^x \frac{dt}{t}, \ \forall x > 0$
2. Methods of integration
   - Integration by parts and substitution
3. Darboux sum convergence theorem
   - If a function is integrable on $[a, b]$, then any sequence of meshes $\{\mathcal{M}_n\}$ such that $\lim_{n\to\infty} \text{gap}(\mathcal{M}_n) = 0$ is an Archimedean sequence of meshes for f on $[a, b]$
4. Riemann sum convergence theorem
   - Expresses the integral of an integrable function as the limit of a sequence of Riemann sums whose gap converges to 0
5. This can be used to find methods to approximate integrals
   - Left, right, midpoint, trapezoid, and Simpson's rules and associated errors

* Taylor polynomials and Taylor series

1. Definition of Taylor polynomials

2. Lagrange remainder theorem
   - Application: use polynomials to estimate (bound) functions

3. Taylor series
   - Taylor series as series of numbers
   - Pointwise convergence of the Taylor series expansion of a function f, using the Lagrange remainder theorem
   - Taylor series (and more generally power series) are uniformly convergent and can be differentiated term by term, for $|x - x_0| < R$ where R is the radius of convergence of the series

4. Taylor series solutions of differential equations
   - Define $\cos(x)$ in terms of its Taylor series, as the solution of $f''(x) + f(x) = 0, \ \forall x \in \mathbb{R}$, with $f(0) = 1$ and $f'(0) = 0$

Sequences of functions

1. Pointwise convergence of sequences of functions
2. Uniform convergence of sequences of functions
3. Properties of uniformly convergent sequences of functions
   - The uniform limit of continuous functions is continuous
   - The uniform limit of integrable functions is integrable
   - Uniformly convergent sequences of differentiable functions
4. Cauchy sequences of functions
   - A sequence of functions converges uniformly if and only if it is uniformly Cauchy