8.2.5 $\begin{cases} \dot{x} = y + \mu x \\ \dot{y} = -x + \mu y - x^2 y \end{cases}$ We first re-write this system in complex form.

Let $z = x + iy$. Then $\dot{z} = \dot{x} + i\dot{y}$

i.e. $\dot{z} = y + \mu x + i(-x + \mu y - x^2 y) = -i(x+iy) + \mu(x+iy) - ix^2 y = -iz + \mu z - ix^2 y$

With $x = \dfrac{z+\bar{z}}{2}$ and $y = \dfrac{z-\bar{z}}{2i}$, the last term reads

$$-ix^2y = -i\left(\frac{z+\bar{z}}{2}\right)^2 \frac{z-\bar{z}}{2i} = -\frac{1}{8}\left(z^2 + 2|z|^2 + \bar{z}^2\right)(z-\bar{z})$$
$$= -\frac{1}{8}\left(z^3 + 2|z|^2 z + |z|^2\bar{z} - |z|^2 z - 2|z|^2\bar{z} - \bar{z}^3\right)$$
$$= -\frac{1}{8}\left(z^3 + |z|^2 z - |z|^2\bar{z} - \bar{z}^3\right)$$

So the equation for z reads $\dot{z} = -iz + \mu z - \frac{1}{8}\left(z^3 + |z|^2 z - |z|^2\bar{z} - \bar{z}^3\right)$.

We "remove" all cubic terms but the term in $|z|^2 z$ by making the following change of variable:

$$z = \tilde{z} + \alpha\tilde{z}^3 + \beta|\tilde{z}|^2\bar{\tilde{z}} + \gamma\bar{\tilde{z}}^3,$$ where α, β and γ are to be chosen later.

Then, $\tilde{z} = z - \alpha\tilde{z}^3 - \beta|\tilde{z}|^2\bar{\tilde{z}} - \gamma\bar{\tilde{z}}^3 = z - \alpha z^3 - \beta|z|^2\bar{z} - \gamma\bar{z}^3 + \text{h.o.t.}$, where h.o.t. stands for terms of order 4 and higher in z and $\tilde{z}$.

The differential equation for $\tilde{z}$ therefore reads

$$\dot{\tilde{z}} = \dot{z} - 3\alpha z^2\dot{z} - \beta\left(\dot{z}\bar{z}^2 + z\,2\dot{\bar{z}}\bar{z}\right) - 3\gamma\bar{z}^2\dot{\bar{z}} + \dot{\overline{\text{h.o.t.}}}$$
$$= -iz + \mu z - \frac{1}{8}\left(z^3 + |z|^2 z - |z|^2\bar{z} - \bar{z}^3\right) - 3\alpha z^2\left(-iz + \mu z + O(3)\right)$$
$$-\beta\bar{z}^2\left(-iz + \mu z + O(3)\right) - 2|z|^2\beta\left(i\bar{z} + \mu\bar{z} + O(3)\right) - 3\gamma\bar{z}^2\left(i\bar{z} + \mu\bar{z} + O(3)\right) + O(4)$$

where $O(n)$ stands for term of order n or higher in z and $\bar{z}$.

So $$\dot{\tilde{z}} = -iz + \mu z - \frac{1}{8}\left(z^3 + |z|^2 z - |z|^2\bar{z} - \bar{z}^3\right) + 3\alpha z^3(i-\mu) + \beta|z|^2\bar{z}(i-\mu)$$
$$-2\beta|z|^2\bar{z}(i+\mu) - 3\gamma\bar{z}^3(i+\mu) + O(4)$$
$$= (-i+\mu)\left(\tilde{z} + \alpha\tilde{z}^3 + \beta|\tilde{z}|^2.\bar{\tilde{z}} + \gamma\bar{\tilde{z}}^3\right) - \frac{1}{8}\left(\tilde{z}^3 + |\tilde{z}|^2\tilde{z} - |\tilde{z}|^2\bar{\tilde{z}} + \bar{\tilde{z}}^3\right)$$
$$+ 3\alpha\tilde{z}^3(i-\mu) + \beta|\tilde{z}|^2\bar{\tilde{z}}(i - \mu - 2i - 2\mu) - 3\gamma\bar{\tilde{z}}^3(i+\mu) + O(4)$$

i.e. $\dot{\tilde{z}} = (-i+\mu)\,\tilde{z} + \alpha\,\tilde{z}^3\left[-i+\mu+3i-3\mu\right] + \beta\,|\tilde{z}|^2\bar{\tilde{z}}\left[-i+\mu-i-3\mu\right]$

$$+\gamma\,\bar{\tilde{z}}^3\left(-i+\mu-3i-3\mu\right) - \frac{1}{8}\left(\tilde{z}^3 + |\tilde{z}|^2\tilde{z} - |\tilde{z}|^2\bar{\tilde{z}} - \bar{\tilde{z}}^2\right) + O(4)$$

$$= (-i+\mu)\,\tilde{z} + \tilde{z}^3\left[\alpha\, 2(i-\mu) - \frac{1}{8}\right] + |\tilde{z}|^2\bar{\tilde{z}}\left[-2\beta(\mu+i) + \frac{1}{8}\right]$$

$$+\bar{\tilde{z}}^3\left[-2\gamma(\mu+2i) + \frac{1}{8}\right] - \frac{1}{8}|\tilde{z}|^2\tilde{z} + O(4)$$

If we choose $\alpha = \dfrac{1}{16(i-\mu)}$, $\beta = \dfrac{1}{16(i+\mu)}$ and $\gamma = \dfrac{1}{16(\mu+2i)}$,

we see that the equation for $\tilde{z}$ reduces to

$$\dot{\tilde{z}} = (-i+\mu)\,z - \frac{1}{8}|\tilde{z}|^2\tilde{z} + O(4)$$

We look for a limit cycle as a solution to $\dot{\tilde{z}} = (-i+\mu)\tilde{z} - \frac{1}{8}|\tilde{z}|^2\tilde{z}$, in the form $\tilde{z} = r e^{i\beta t}$ $r,\beta \in \mathbb{R}$, $r > 0$.

Then $\dot{\tilde{z}} = i\beta r e^{i\beta t}$

$$= (-i+\mu)\, r e^{i\beta t} - \frac{1}{8} r^3 e^{i\beta t}$$

so $\begin{cases} 0 = \mu r - \frac{1}{8} r^3 \\ \beta = -1 \end{cases} \Rightarrow \begin{cases} r^2 = 8\mu \\ \beta = -1 \end{cases}$.

The amplitude r scales like the square root of the control parameter

A quick exploration with PPLANE gives the following approximate resu

μ	approximate value of r	approximate value of r^2
0.01	0.28	0.08
0.02	0.40	0.16
0.03	0.48	0.23
0.05	0.62	0.38

Fitting these points with a line that goes through the origin in the (μ, r^2) plane gives a slope of about $7.67 \simeq 8$, which illustrates that the normal form works well, especially near threshold