8.2.7 $\begin{cases}\dot x = \mu x + y - x^2 \\ \dot y = -x + \mu y + 2x^2\end{cases}$

Let $z = x+iy$. Then, $\dot z = \dot x + i\dot y = \mu x + y - x^2 + i(-x+\mu y + 2x^2)$

$$= -i(x+iy) + \mu(x+iy) - x^2(1-2i) = -iz + \mu z - x^2(1-2i)$$

With $x = \frac{z+\bar z}{2}$, we get

$$\dot z = -iz + \mu z - \left(\frac{z+\bar z}{2}\right)^2(1-2i) = -iz+\mu z - \frac{1-2i}{4}(z^2 + 2|z|^2 + \bar z^2).$$

We want to "remove" the term in $|z|^2$. To this end, we let

$$z = \tilde z + \alpha|\tilde z|^2.$$

Then $\tilde z = z - \alpha|\tilde z|^2 = z - \alpha\tilde z\bar{\tilde z} = z - \alpha(z-\alpha\tilde z\bar{\tilde z})(\bar z - \bar\alpha \bar{\tilde z}\tilde z)$

$$= z - \alpha\Big[|z|^2 - \alpha\bar z(z-\alpha|\tilde z|^2)(\bar z - \bar\alpha|\tilde z|^2) - \bar\alpha z(\bar z - \bar\alpha|\tilde z|^2)(z-\alpha|\tilde z|^2) + O(|\tilde z|^4)\Big]$$

$$= z - \alpha\left[|z|^2 - \alpha|z|^2\bar z - \bar\alpha|z|^2 z + O(|\tilde z|^4)\right]$$

$$= z - \alpha|z|^2 + \alpha^2|z|^2\bar z + |\alpha|^2|z|^2 z + O(4),$$

where $O(4)$ represents terms of order 4 in z and $\bar z$ (or $\tilde z$ and $\bar{\tilde z}$).

We now write an equation for the new variable $\tilde z$.

$$\dot{\tilde z} = \dot z - \alpha\dot z\bar z - \alpha z\dot{\bar z} + \alpha^2(\dot z\bar z^2 + z2\bar z\dot{\bar z}) + |\alpha|^2(2z\dot z\bar z + z^2\dot{\bar z}) + O(4)$$

$$= (\mu-i)z - \frac{1-2i}{4}(z^2+2|z|^2+\bar z^2) - \alpha\bar z\left((\mu-i)z - \frac{1-2i}{4}(z^2+2|z|^2+\bar z^2)\right)$$

$$-\alpha z\left((\mu+i)\bar z - \frac{1+2i}{4}(\bar z^2 + 2|z|^2 + z^2)\right) + \alpha^2\left((\mu-i)z\bar z^2 + 2|z|^2(\mu+i)\bar z\right)$$

$$+|\alpha|^2\left(2|z|^2(\mu-i)z + z^2(\mu+i)\bar z\right) + O(4)$$

$$= (\mu-i)\left[\tilde z + \alpha|\tilde z|^2\right] - \frac{1-2i}{4}\left[(\tilde z+\alpha|\tilde z|^2)^2 + 2(\tilde z+\alpha|\tilde z|^2)(\bar{\tilde z}+\bar\alpha|\tilde z|^2) + (\bar{\tilde z}+\bar\alpha|\tilde z|^2)^2\right]$$

$$-\alpha(\mu-i)(\bar{\tilde z}+\bar\alpha|\tilde z|^2)(\tilde z+\alpha|\tilde z|^2) + \alpha\frac{1-2i}{4}(|\tilde z|^2\tilde z + 2|\tilde z|^2\bar{\tilde z} + \bar{\tilde z}^3)$$

$$-\alpha(\mu+i)(\tilde z+\alpha|z|^2)(\bar{\tilde z}+\bar\alpha|\tilde z|^2) + \alpha\frac{1+2i}{4}(|\tilde z|^2\bar{\tilde z} + 2|\tilde z|^2\tilde z + \tilde z^3)$$

$$+\alpha^2(3\mu+i)|\tilde z|^2\bar{\tilde z} + |\alpha|^2(3\mu-i)|\tilde z|^2\tilde z + O(4)$$

$$\dot{\tilde{z}} = (\mu - i)\left(\tilde{z} + \alpha|\tilde{z}|^2\right) - \frac{1-2i}{4}\Big[\tilde{z}^2 + 2\alpha|\tilde{z}|^2\tilde{z} + 2\left(|\tilde{z}|^2 + \alpha|\tilde{z}|^2\bar{\tilde{z}} + \bar{\alpha}|\tilde{z}|^2\tilde{z}\right) + \bar{\tilde{z}}^2 + 2\bar{\alpha}|\tilde{z}|^2\bar{\tilde{z}}\Big]$$

$$- 2\alpha\mu\left(|\tilde{z}|^2 + \bar{\alpha}|\tilde{z}|^2\tilde{z} + \alpha|\tilde{z}|^2\bar{\tilde{z}}\right) + \alpha\frac{1-2i}{4}\left(|\tilde{z}|^2\tilde{z} + 2|\tilde{z}|^2\bar{\tilde{z}} + \bar{\tilde{z}}^3\right)$$

$$+ \alpha\frac{1+2i}{4}\left(|\tilde{z}|^2\bar{\tilde{z}} + 2|\tilde{z}|^2\tilde{z} + \tilde{z}^3\right) + \alpha^2(3\mu + i)|\tilde{z}|^2\bar{\tilde{z}} + |\alpha|^2(3\mu - i)|\tilde{z}|^2\tilde{z} + O(4)$$

$$= (\mu - i)\tilde{z} - \frac{1-2i}{4}\left(\tilde{z}^2 + \bar{\tilde{z}}^2\right) + |\tilde{z}|^2\left(\alpha(\mu - i) - \frac{1-2i}{2} - 2\alpha\mu\right)$$

$$+ |\tilde{z}|^2\tilde{z}\left[-\alpha\frac{1-2i}{2} - \bar{\alpha}\frac{1-2i}{2} - 2\mu|\alpha|^2 + \alpha\frac{1-2i}{4} + \alpha\frac{1+2i}{2} + |\alpha|^2(3\mu - i)\right]$$

$$+ |\tilde{z}|^2\bar{\tilde{z}}\left[-\bar{\alpha}\frac{1-2i}{2} - \alpha\frac{1-2i}{2} - 2\mu\alpha^2 + \alpha\frac{1-2i}{2} + \alpha\frac{1+2i}{4} + \alpha^2(3\mu + i)\right]$$

$$+ \bar{\tilde{z}}^3\left[\alpha\frac{1-2i}{4}\right] + \alpha\frac{1+2i}{4}\tilde{z}^3 + O(4)$$

$$= (\mu - i)\tilde{z} - \frac{1-2i}{4}\left(\tilde{z}^2 + \bar{\tilde{z}}^2\right) + |\tilde{z}|^2\left[-\alpha(\mu + i) - \frac{1-2i}{2}\right] + \alpha\frac{1-2i}{4}\bar{\tilde{z}}^3 + \alpha\frac{1+2i}{4}\tilde{z}^3$$

$$+ |\tilde{z}|^2\tilde{z}\left[\frac{\alpha}{4}(-2 + 4i + 1 - 2i + 2 + 4i) - \frac{\bar{\alpha}}{2}(1 - 2i) + |\alpha|^2(\mu - i)\right]$$

$$+ |\tilde{z}|^2\bar{\tilde{z}}\left[-\frac{\bar{\alpha}}{2}(1 - 2i) - \frac{\alpha}{4}(2 - 4i - 2 + 4i - 1 - 2i) + \alpha^2(\mu + i)\right] + O(4)$$

$$= (\mu - i)\tilde{z} - \frac{1-2i}{4}\left(\tilde{z}^2 + \bar{\tilde{z}}^2\right) + |\tilde{z}|^2\left[-\alpha(\mu + i) - \frac{1-2i}{2}\right] + \alpha\frac{1-2i}{4}\bar{\tilde{z}}^3 + \alpha\frac{1+2i}{4}\tilde{z}^3$$

$$+ |\tilde{z}|^2\tilde{z}\left[\frac{\alpha}{4}(1 + 6i) - \frac{\bar{\alpha}}{2}(1 - 2i) + |\alpha|^2(\mu - i)\right] + |\tilde{z}|^2\bar{\tilde{z}}\left[-\frac{\bar{\alpha}}{2}(1 - 2i) - \frac{\alpha}{4}(-1 - 2i) + \alpha^2(\mu + i)\right] + O(4)$$

If we now choose $\alpha = -\frac{1-2i}{2(\mu + i)}$, we can remove the term in $|\tilde{z}|^2$ from the equation. We however see that this transformation generated many cubic terms. We can similarly remove the terms in $\tilde{z}^2$ and $\bar{\tilde{z}}^2$, which will also generate cubic terms. Further near-identity changes of variable will allow us to remove all of the cubic terms, except the term in $|\tilde{z}|^2\tilde{z}$, which will appear in the normal form of the Hopf bifurcation for this system.