8.2.8 Normal form

$$\begin{cases} \dot{x} = x\,[x(1-x) - y] \\ \dot{y} = y\,(x-a) \end{cases} \qquad \begin{matrix} x, y \geq 0 \\ a \geq 0 \end{matrix}$$

The fixed point with coordinates $(a, a-a^2)$ undergoes a Hopf bifurcation at $a = a_c = 1/2$.

1. Shift coordinates so that the fixed point $(a, a-a^2)$ is at the origin of the new coordinate system.

Let $\quad x = a + X \qquad$ i.e. $X = x - a$

$\qquad\;\; y = a - a^2 + Y \qquad$ i.e. $Y = y - a + a^2$

Then,
$$\begin{aligned}
\dot{X} = \dot{x} &= x^2(1-x) - xy = (a+X)^2(1-a-X) - (a+X)(a-a^2+Y) \\
&= a^2 + 2aX + X^2 - a^3 - 2a^2X - aX^2 - a^2X - 2aX^2 - X^3 - a^2 + a^3 - aY - aX + a^2X - XY \\
&= X(2a - 2a^2 - a^2 - a + a^2) + Y(-a) + X^2(1 - a - 2a) - X^3 - XY \\
&= (a - 2a^2)X - aY + (1-3a)X^2 - X^3 - XY
\end{aligned}$$

$$\dot{Y} = \dot{y} = y\,(x-a) = (a - a^2 + Y)\,X = (a-a^2)X + XY$$

So the new system reads
$$\begin{cases} \dot{X} = (a-2a^2)X - aY + (1-3a)X^2 - X^3 - XY \\ \dot{Y} = (a-a^2)X + XY \end{cases}$$

2. Re-write the system in complex form

Note that the Jacobian of the system for X & Y is $J_a(0,0) = \begin{pmatrix} a-2a^2 & -a \\ a-a^2 & 0 \end{pmatrix}$, as expected from the classification of the fixed point $(a, a-a^2)$ of the original system.

When $a = a_c$, $\quad J_{\frac{1}{2}}(0,0) = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{4} & 0 \end{pmatrix}$ and its eigenvalues are $\lambda = \pm i \frac{1}{2\sqrt{2}}$.

We will now re-write the (X,Y) system in the basis of eigenvectors of $J_{\frac{1}{2}}(0,0)$, at the bifurcation point, i.e. when $a = a_c = \frac{1}{2}$.

An eigenvector with eigenvalue $\frac{i}{2\sqrt{2}}$ satisfies

$$\begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{4} & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{i}{2\sqrt{2}} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad \text{i.e.} \quad -\frac{1}{2}\beta = \frac{i}{2\sqrt{2}}\alpha, \quad \text{i.e.} \quad i\alpha = -\sqrt{2}\beta .$$

So for instance $\gamma = \begin{pmatrix} -\sqrt{2} \\ i \end{pmatrix}$ is an eigenvector of $J_{\frac{1}{2}}(0,0)$ with eigenvalue $\frac{i}{2\sqrt{2}}$.

We now define a complex variable z such that

$$\begin{pmatrix} X \\ Y \end{pmatrix} = z \begin{pmatrix} -\sqrt{2} \\ i \end{pmatrix} + \bar{z} \begin{pmatrix} -\sqrt{2} \\ -i \end{pmatrix} = \begin{pmatrix} -\sqrt{2}\,(z+\bar{z}) \\ i\,(z-\bar{z}) \end{pmatrix}$$

and derive an equation for the z-variable, at the bifurcation point.

$$\left.\begin{aligned} z+\bar{z} &= -\frac{X}{\sqrt{2}} \\ z-\bar{z} &= -iY \end{aligned}\right\} \Rightarrow z = -\frac{1}{2}\left(\frac{X}{\sqrt{2}} + iY\right) \quad \text{and} \quad \dot{z} = \frac{-1}{2\sqrt{2}}\dot{X} - \frac{i}{2}\dot{Y}.$$

At the bifurcation $(a = a_c = \frac{1}{2})$, we have

$$\begin{cases} \dot{X} = 0\cdot X - \frac{1}{2}Y - \frac{1}{2}X^2 - X^3 - XY = -\frac{1}{2}Y - \frac{1}{2}X^2 - X^3 - XY \\ \dot{Y} = \frac{1}{4}X + XY \end{cases}$$

so that $\dot{z} = -\frac{1}{2\sqrt{2}}\left(-\frac{1}{2}Y - \frac{1}{2}X^2 - X^3 - XY\right) - \frac{i}{2}\left(\frac{1}{4}X + XY\right)$

$$\begin{aligned}
\text{i.e. } \dot{z} &= \frac{1}{4\sqrt{2}}\left(\frac{z-\bar{z}}{-i} + 2(z+\bar{z})^2 - 4\sqrt{2}(z+\bar{z})^3 - 2\sqrt{2}(z+\bar{z})\frac{z-\bar{z}}{-i}\right) \\
&\quad - \frac{i}{8}\left(-\sqrt{2}(z+\bar{z}) + 4\sqrt{2}(z+\bar{z})\frac{z-\bar{z}}{i}\right) \\
&= \frac{1}{4\sqrt{2}}\left(i(z-\bar{z}) + 2(z+\bar{z})^2 - 4\sqrt{2}(z+\bar{z})^3 - 2i\sqrt{2}(z^2-\bar{z}^2)\right) + \frac{i}{4\sqrt{2}}\left(z+\bar{z}+4i(z^2-\bar{z}^2)\right) \\
&= \frac{1}{4\sqrt{2}}\left[z(i+i) + \bar{z}(-i+i) + z^2(2-2i\sqrt{2}-4) + |z|^2(4) + \bar{z}^2(2+2i\sqrt{2}+4) - 4\sqrt{2}(z+\bar{z})^3\right] \\
&= \frac{1}{4\sqrt{2}}\left[2iz + 2(-i\sqrt{2}-1)z^2 + 4|z|^2 + 2(3+i\sqrt{2})\bar{z}^2 - 4\sqrt{2}(z+\bar{z})^3\right] \\
&= \frac{1}{2\sqrt{2}}\left[iz + (-i\sqrt{2}-1)z^2 + 2|z|^2 + (3+i\sqrt{2})\bar{z}^2 - 2\sqrt{2}(z+\bar{z})^3\right]
\end{aligned}$$

This is the equation we would use to derive the normal form of the corresponding Hopf bifurcation.

If instead we want to use the criterion of problem # 8.2.12, then we need to write an equation for the real part of z and an equation for its imaginary part.

Let $p = \mathrm{Re}(z) = \dfrac{z+\bar{z}}{2} = \dfrac{-x}{2\sqrt{2}}$ and $q = \mathrm{Im}(z) = \dfrac{z-\bar{z}}{2i} = \dfrac{-y}{2}$.

We have

$$\dot{p} = \frac{-\dot{x}}{2\sqrt{2}} = \frac{1}{2\sqrt{2}}\left(\frac{1}{2}y + \frac{1}{2}x^2 + x^3 + xy\right) = \frac{1}{2\sqrt{2}}\left(-q + 4p^2 - 8\cdot 2\sqrt{2}\,p^3 + 2\sqrt{2}\,p\cdot 2q\right)$$

$$= \frac{1}{2\sqrt{2}}\left(-q + 4p^2 - 16\sqrt{2}\,p^3 + 4\sqrt{2}\,pq\right) = \frac{-q}{2\sqrt{2}} + \sqrt{2}\,p^2 - 8p^3 + 2pq$$

and

$$\dot{q} = \frac{-\dot{y}}{2} = \frac{-x}{8} - \frac{1}{2}xy = \frac{2\sqrt{2}\,p}{8} - \frac{1}{2}\left(-2\sqrt{2}\,p\right)2q = \frac{p}{2\sqrt{2}} + 2\sqrt{2}\,pq$$

So the system in the p, q variables reads

$$\begin{cases} \dot{p} = \dfrac{-q}{2\sqrt{2}} + \sqrt{2}\,p^2 - 8p^3 + 2pq = -\omega q + f(p,q) \\ \dot{q} = \dfrac{p}{2\sqrt{2}} - 2\sqrt{2}\,pq = \omega p + g(p,q) \end{cases}$$

with $\omega = \dfrac{1}{2\sqrt{2}}$, $f(p,q) = \sqrt{2}\,p^2 - 8p^3 + 2pq$.

$g(p,q) = -2\sqrt{2}\,pq$

We have $f_p = 2\sqrt{2}\,p - 24p^2 + 2q$; $f_{pp} = 2\sqrt{2} - 48p$; $f_{ppp} = -48$; $f_{pq} = 2$; $f_q = 2p$;

$g_p = -2\sqrt{2}\,q$; $g_q = -2\sqrt{2}\,p$

So the coefficient of the cubic term in the normal form has real part a, with

$$16a = -48 + 2\sqrt{2}\left[2\left(2\sqrt{2} - 48\cdot 0\right) + 2\sqrt{2}\,(0) - \left(2\sqrt{2} - 48\cdot 0\right)(0) + 0\right]$$

$$= -48 + 2\sqrt{2}\cdot 4\sqrt{2} = -48 + 16 = -32$$

So $\boxed{a = -2}$ and the bifurcation is supercritical, as was concluded from the numerical exploration of problem 8.2.8.