Normal form of the Hopf bifurcation

These notes review the derivation of the normal form of the Hopf bifurcation by means of near-identity changes of variables, on the example of the following system:

$$\begin{aligned} \frac{dx}{dt} &= \mu x + y \\ \frac{dy}{dt} &= -x + \mu y - x^2 y \end{aligned} \tag{1}$$

The origin is a fixed point of (1), and it undergoes a Hopf bifurcation at $\mu = 0$. Since the Jacobian of (1) at the origin, evaluated at $\mu = 0$, is anti-diagonal, we can immediately introduce a complex variable

$$z = x + iy,$$

and derive an equation for the dynamics of z. This equation reads

$$\begin{aligned} \frac{dz}{dt} &= \frac{dx}{dt} + i\frac{dy}{dt} \\ &= \mu x + y + i(-x + \mu y - x^2 y) = \mu(x+iy) - i(x+iy) - ix^2 y \\ &= \mu z - iz - i\left(\frac{z+\bar{z}}{2}\right)^2 \frac{z-\bar{z}}{2} = \mu z - iz - \frac{1}{8}\left(z^2 + 2|z|^2 + \bar{z}^2\right)(z - \bar{z}) \\ &= \mu z - iz - \frac{1}{8}\left(z^3 + 2|z|^2 z + |z|^2\bar{z} - |z|^2 z - 2|z|^2\bar{z} - \bar{z}^3\right) \\ &= \mu z - iz - \frac{1}{8}\left(z^3 + |z|^2 z - |z|^2\bar{z} - \bar{z}^3\right), \end{aligned}$$

where we have used the fact that

$$x = \frac{z+\bar{z}}{2} \qquad \text{and} \qquad y = \frac{z-\bar{z}}{2i}.$$

The above equation contains linear and cubic terms in z and $\bar{z}$. We are going to make a *near-identity* change of variables to "remove" some of the cubic terms. This transformation will generate nonlinear terms of order higher than 3. In what follows, we illustrate this process by eliminating only one of the terms, of the form $z^p\bar{z}^q$, with $p+q=3$. To do so, we first make the near-identity change of variable

$$z = \tilde{z} + \alpha\, \tilde{z}^p\bar{\tilde{z}}^q, \tag{2}$$

where α is an unknown, *a priori complex coefficient*, to be determined later. This relation can be inverted as follows:

$$\tilde{z} = z - \alpha\, \tilde{z}^p\bar{\tilde{z}}^q = z - \alpha\, \left(z - \alpha\, \tilde{z}^p\bar{\tilde{z}}^q\right)^p \left(\overline{z - \alpha\, \tilde{z}^p\bar{\tilde{z}}^q}\right)^q = z - \alpha z^p \bar{z}^q + \text{h.o.t.}, \tag{3}$$

where h.o.t. stands for terms of order 4 and higher in z and $\bar{z}$. Then,

$$\begin{aligned}
\frac{d\tilde{z}}{dt} &= \frac{dz}{dt} - \alpha\, p\, z^{p-1}\frac{dz}{dt}\,\bar{z}^q - \alpha\, z^p\, q\, \bar{z}^{q-1}\frac{d\bar{z}}{dt} + \frac{d}{dt}(\text{h.o.t.}) \\
&= \mu z - iz - \frac{1}{8}\left(z^3 + |z|^2 z - |z|^2\bar{z} - \bar{z}^3\right) \\
&\quad -\alpha\, p\, z^{p-1}\left(\mu z - iz + \mathcal{O}(3)\right)\bar{z}^q - \alpha\, z^p\, q\, \bar{z}^{q-1}\left(\mu\bar{z} + i\bar{z} + \mathcal{O}(3)\right) + \text{h.o.t.} \\
&= \mu z - iz - \frac{1}{8}\left(z^3 + |z|^2 z - |z|^2\bar{z} - \bar{z}^3\right) - \alpha(\mu - i)p\, z^p\bar{z}^q - \alpha(\mu + i)q\, z^p\bar{z}^q + \text{h.o.t.} \\
&= \mu\left(\tilde{z} + \alpha\,\tilde{z}^p\bar{\tilde{z}}^q\right) - i\left(\tilde{z} + \alpha\,\tilde{z}^p\bar{\tilde{z}}^q\right) - \frac{1}{8}\left(\tilde{z}^3 + |\tilde{z}|^2\tilde{z} - |\tilde{z}|^2\bar{\tilde{z}} - \bar{\tilde{z}}^3\right) \\
&\quad -\alpha\left(p(\mu - i) + q(\mu + i)\right)\tilde{z}^p\bar{\tilde{z}}^q + \text{h.o.t.} \\
&= (\mu - i)\tilde{z} - \frac{1}{8}\left(\tilde{z}^3 + |\tilde{z}|^2\tilde{z} - |\tilde{z}|^2\bar{\tilde{z}} - \bar{\tilde{z}}^3\right) - \alpha\left(-\mu + i + p(\mu - i) + q(\mu + i)\right)\tilde{z}^p\bar{\tilde{z}}^q \\
&\quad +\text{h.o.t.}
\end{aligned}$$

In the above calculation, $\mathcal{O}(3)$ denotes cubic terms in z and $\bar{z}$, and h.o.t. refers to terms of order 4 and higher, in z and $\bar{z}$ or $\tilde{z}$ and $\bar{\tilde{z}}$. Also note that we first expressed $d\tilde{z}/dt$ in terms of z and $\bar{z}$, and then used the expression of z as a function of $\tilde{z}$ and $\bar{\tilde{z}}$, to obtain an equation, correct to order 3, that only involves $\tilde{z}$ and $\bar{\tilde{z}}$.

We now see that in order to get an equation that does not contain the nonlinear term $\tilde{z}^p\bar{\tilde{z}}^q$, with $p + q = 3$, we need to choose α so that this term cancels out from the equation for $\tilde{z}$. You can check that all of the cubic terms can be removed, except the term in $|\tilde{z}|^2\tilde{z}$. Indeed, to remove such a term, we would have to choose α such that

$$-\alpha\left(-\mu + i + p(\mu - i) + q(\mu + i)\right) - \frac{1}{8} = 0,$$

with $p = 2$ and $q = 1$. This reads

$$-\alpha\left(-\mu + i + 2\mu - 2i + \mu + i\right) - \frac{1}{8} = 0,$$

i.e.

$$-2\mu\alpha - \frac{1}{8} = 0,$$

whose solution, $\alpha = \dfrac{-1}{16\mu}$ diverges at the bifurcation, when $\mu = 0$. In this case, the near-identity change of variable necessary to remove the term in $|z|^2 z$ is not defined across the bifurcation, and therefore cannot be done.

Once we have removed the 3 other cubic terms, we are left with the normal form for the Hopf bifurcation of system (1), which reads

$$\frac{d\tilde{z}}{dt} = (\mu - i)\tilde{z} - \frac{1}{8}|\tilde{z}|^2\tilde{z} + \text{h.o.t.} \tag{4}$$

Equation (4) shows that the bifurcation is supercritical (since the coefficient of the cubic term is negative), and that the frequency of oscillations of the limit cycle that exists above the bifurcation does not depend on its amplitude (since the imaginary part of the coefficient of the cubic term is zero).

By letting $u = \frac{1}{\sqrt{8}}\tilde{z}e^{it}$, one obtains

$$\frac{du}{dt} = \mu u - |u|^2 u + \text{h.o.t.},$$

which is the scaled normal form for a supercritical Hopf bifurcation.