Topics covered in MATH 454

1 Definition and attributes of a dynamical system

- A dynamical system is often characterized in terms of the following attributes:
  - Its dimension;
  - Its linear or nonlinear nature;
  - Its autonomous or non-autonomous character.
- We only consider dynamical systems which are continuously differentiable. This implies that solutions exist and are uniquely determined by initial conditions.
- We assume that the right-hand side of the dynamical system has real coefficients and that the dependent variables are therefore always real.

2 Special solutions

2.1 Fixed points

- Definition of a fixed point.
- How to find all of the fixed points of a dynamical system.

2.2 Stability of a fixed point

2.2.1 One-dimensional systems

For one-dimensional systems, one can use the right-hand side of the equation to determine the *nonlinear* stability of the fixed point. *Linear stability* is determined by the sign of the derivative of the right-hand side calculated at the fixed point.

2.2.2 Two-dimensional systems

- For two-dimensional systems, we start by linearizing the system about a fixed point. The trace and the determinant of the Jacobian allows us to classify the fixed point and to determine its *linear stability* properties.
- The *Hartman-Grobman theorem* tells us that if a fixed point is hyperbolic (i.e. if all of the eigenvalues of the Jacobian of the linearization about the fixed point have non-zero real parts), then the nature of the fixed point does not change when *nonlinear* effects are taken into account.

- More work needs to be done in order to determine the *nonlinear stability* properties of non-hyperbolic fixed points.

2.3 Limit cycles and periodic orbits

2.3.1 Existence and non-existence of limit cycles and periodic orbits

We discussed ways of proving the existence of limit cycles, or of ruling out their presence.

- The *Poincaré-Bendixson theorem* can be used to prove the existence of a closed curve in some region of the phase plane.
- Gradient systems do not have periodic orbits.
- If a Liapunov function exists for a particular dynamical system, then this system has no periodic orbits.
- Periodic orbits can also be ruled out by application of *Dulac's criterion*.

2.3.2 Different types of oscillatory behaviors

We discussed two types of oscillatory behaviors.

- Relaxation oscillations, typically seen in *excitable systems*.
- Weakly nonlinear oscillations, which can be analyzed by means of the *Poincaré-Lindstedt method*.

3 Phase portraits

Two-dimensional phase portraits can often be qualitatively described by

- Identifying the fixed points and classifying them.
- Plotting the stable and unstable manifolds of saddle points.
- Establishing the presence of limit cycles.
- Looking at the direction of the flow on particular curves (for instance axes or null-clines).
- Using *index theory*.
- Plotting a few representative trajectories, on the basis of the above information.

There are however special systems whose phase portraits can easily be plotted. These are

- Conservative systems.
- Reversible systems.

In particular, it is often easy to prove the existence of *nonlinear centers* in such systems.

4 Bifurcations

4.1 Different types of bifurcations

Bifurcations that may occur in dynamical systems of dimension one or higher are

- The saddle-node bifurcation;
- The transcritical bifurcation;
- The pitchfork bifurcation, which can be supercritical or subcritical.

Hopf bifurcations may only occur in systems of dimension two or higher.

4.2 Normal forms

Normal forms are *minimal* equations that capture the nature of a particular bifurcation (the word *minimal* is used in the sense that only the relevant linear and nonlinear terms appear in the normal form).

4.2.1 One-dimensional systems

For one-dimensional systems, identifying bifurcations and finding their normal forms involves the following steps:

- Plot the bifurcation diagram, showing the branches of fixed points as functions of the control parameter.
- Identify each bifurcation point (bifurcations typically occur when two branches of fixed points collide). Determine the nature of each bifurcation based on the *local* characteristics of the bifurcation diagram.
- For each bifurcation, change variables so that the bifurcation occurs at $x = 0$ and when the control parameter crosses zero.
- Taylor-expand the system in these new variables and only keep the relevant linear and nonlinear terms.
- Re-scale time and x to set most coefficients in the normal form to 1 or -1.
- Use the normal form to determine/confirm the stability of the fixed points before and after the bifurcation, and in the case of pitchfork bifurcations, to decide if the bifurcation is supercritical or subcritical.

4.2.2 Two-dimensional systems

For two-dimensional systems, saddle-node, transcritical and pitchfork bifurcations are described and characterized by one-dimensional normal forms. You should be able to recognize a particular bifurcation based on the corresponding changes that occur in the phase plane.

A *Hopf bifurcation* occurs when the two complex-conjugate eigenvalues of the Jacobian of the linearization about a fixed point cross the imaginary axis. The bifurcation is supercritical if a limit cycle exists after the fixed point has lost stability, and subcritical if the limit cycle exists before the fixed point has lost stability. The normal form of the Hopf bifurcation is an equation for a single complex variable, and has complex coefficients. The steps to derive such a normal form are as follows:

- Change variables in the original two-dimensional system so that the bifurcation occurs at $x = y = 0$ and when the control parameter crosses zero.

- Re-write the resulting system in the basis of eigenvectors of the Jacobian of the linearization about the fixed point, evaluated at the bifurcation. Use one complex coordinate variable (z), instead of two real variables.

- Taylor-expand the right-hand side of the differential equation for z at least to order three in powers of z and $\bar{z}$.

- Make near-identity changes of variables to remove the quadratic terms (if any) from the right-hand side of the equation for z. This operation generates cubic terms.

- Make near-identity changes of variables to show that all of the cubic terms can be removed from the equation for z, except the term in $|z|^2 z$. The resulting equation is the normal form of the Hopf bifurcation.

- Re-scale time and z to set some of the coefficients in the normal form to 1 or -1.

- Use the normal form to determine the amplitude of the limit cycle as a function of the control parameter, and to describe how the frequency of oscillations depends on the amplitude of the limit cycle. The normal form can also be used to study the stability of the limit cycle.