MATH 454
Quiz #1

Name: ______________________

Note: Show all of your work and explain what you are doing.

For each (a)-(e), find an equation $\frac{dx}{dt} = f(x)$ with the stated properties, or if there are no examples, explain why not. (In all cases, assume that f is a smooth function.)

(a) Every real number is a fixed point.

(b) Every integer is a fixed point, and there are no others.

(c) There are precisely three fixed points, and all of them are stable.

(d) There are no fixed points.

(e) There are precisely 100 fixed points.

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Quiz #2 Name: ______________________

Note: Show all of your work and explain what you are doing.

In parts (a)-(b), let $V(x)$ be the potential, in the sense that $\frac{dx}{dt} = -\frac{dV}{dx}$. Sketch the potential as a function of r. Be sure to show all the qualitatively different cases, including bifurcation values of r.

(a) (Saddle-node) $\frac{dx}{dt} = r - x^2$

(b) (Transcritical) $\frac{dx}{dt} = r\,x - x^2$

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Quiz #3 Name: ________________________

Note: Show all of your work and explain what you are doing.

Find the eigenvalues and eigenvectors of the following matrix

$$B = \begin{bmatrix} -1 & -4 & -4 \\ 2 & 4 & 2 \\ 0 & 1 & 3 \end{bmatrix}.$$

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Note: Show all of your work and explain what you are doing.

Find the general solution of the following system of differential equations,

$$\frac{dX}{dt} = CX, \qquad X \in \mathrm{R}^3 \qquad \text{and} \qquad C = \begin{bmatrix} 0 & -2 & -2 \\ 3 & 6 & 4 \\ -2 & -3 & -1 \end{bmatrix}.$$

You may find the following information useful:

- The eigenvalues of the matrix C are $\lambda = 2$ with multiplicity 2, and $\lambda = 1$ with multiplicity one.
- The eigenspace for $\lambda = 1$ is spanned by $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$.
- The (genuine) eigenspace for $\lambda = 2$ is spanned by $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

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Note: Show all of your work and explain what you are doing.

Plot the phase portrait and classify the fixed point of the following linear system. If the eigenvectors are real, indicate them in your sketch.

$$\begin{cases} \dfrac{dx}{dt} = 5x + 2y \\ \dfrac{dy}{dt} = -17x - 5y \end{cases}$$

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Note: Show all of your work and explain what you are doing.

Find the fixed points of the following system, classify them, sketch the neighboring trajectories, and try to fill in the rest of the phase portrait.

$$\begin{cases} \dfrac{dx}{dt} = x - y \\ \dfrac{dy}{dt} = x^2 - 4 \end{cases}$$

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Note: Show all of your work and explain what you are doing.

(a) Show that the Duffing equation, $\frac{d^2x}{dt^2} + x + \varepsilon x^3 = 0$ has a nonlinear center at the origin for all $\varepsilon > 0$.

(b) If $\varepsilon < 0$, show that all trajectories near the origin are closed. What about trajectories that are far from the origin?

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Note: Show all of your work and explain what you are doing.

Consider the family of linear systems $\begin{cases} \dfrac{dx}{dt} = x\cos(\alpha) - y\sin(\alpha) \equiv f(x,y) \\ \dfrac{dy}{dt} = x\sin(\alpha) + y\cos(\alpha) \equiv g(x,y) \end{cases}$, where α is a parameter that runs over the range $0 \le \alpha \le \pi.$ Let C be a simple closed curve that does not pass through the origin.

(a) Using the formula $I_c = \dfrac{1}{2\pi}\oint_C \dfrac{f\ dg - g\ df}{f^2 + g^2}$, show that I_C is *independent* of α.

(b) Let C be a circle centered at the origin. Compute I_C explicitly by evaluating the integral for any convenient choice of α.

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Quiz #9 Name: ______________________

Note: Show all of your work and explain what you are doing.

Consider the dynamical system $\begin{cases} \dfrac{dx}{dt} = x - y - x\left(x^2 + 5y^2\right) \\ \dfrac{dy}{dt} = x + y - y\left(x^2 + y^2\right) \end{cases}$.

(a) Using $r\,\dot{r} = x\,\dot{x} + y\,\dot{y}$, find the equation for $\dot{r}$ in polar coordinates.

(b) Determine the circle of maximum radius, r_1, centered on the origin, such that all trajectories have a radially *outward* component on it.

(c) Determine the circle of minimum radius, r_2, centered on the origin, such that all trajectories have a radially *inward* component on it.

(d) Given that there are no fixed points in the trapping region $r_1 \le r \le r_2$, prove that the system has a limit cycle somewhere in this region.

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Quiz #10 Name: ________________________

Note: Show all of your work and explain what you are doing.

Show that the system $\begin{cases} \dfrac{dx}{dt} = y - x^3 \\ \dfrac{dy}{dt} = -x - y^3 \end{cases}$ has no closed orbits, by constructing a Liapunov function $V = a\,x^2 + b\,y^2$ with suitable a,b.

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Quiz #11 Name: ________________________

Note: Show all of your work and explain what you are doing.

Odell (1980) considered the system $\begin{cases} \dfrac{dx}{dt} = x\left[x(1-x) - y\right] \\ \dfrac{dy}{dt} = y(x-a) \end{cases}$, where $x \geq 0$ is the dimensionless population of the prey, $y \geq 0$ is the dimensionless population of the predator, and $a \geq 0$ is a control parameter.

(a) Classify the fixed point $(a, a - a^2)$ for $a < 1$.

(b) Show that a Hopf bifurcation occurs at $a_c = ½$. Is it subcritical or supercritical?

(c) Estimate the frequency of limit cycle oscillations for a near the bifurcation.