Many-body adiabatic theory

Sven Bachmann

with Martin Fraas and Wojciech De Roeck

Department of Mathematics
The University of British Columbia

Arizona School of Analysis and Mathematical Physics
UA Tucson, March 2018
‘Would you like to know what fascinates me in science? I find the ultimate poetry in it: in mathematics, the exhilarating vertigo of numbers; with astronomy, the mysterious whisper of the universe. But please, don’t mention truth!’

Amin Maalouf
Part 1. The adiabatic principle
Part 2. Locality in quantum spin systems
Part 3. The many-body adiabatic theorem
Part 4. Linear response theory

References:

- *Elementary Exponential Error Estimates for the Adiabatic Approximation*

- *Adiabatic Theorem for Quantum Spin Systems*

- *The adiabatic theorem and linear response theory for extended quantum systems*
Many-body quantum systems

The general themes of the course

- Use **locality** properties, a.k.a. propagation estimates, of quantum systems to go beyond the noninteracting approximation in condensed matter physics
- Combine the **long time limit** and the **thermodynamic limit** for non-autonomous dynamics

Common issues to many current topics in **quantum information theory**, **quantum control** and **quantum statistical mechanics**
PART I

Adiabatic Principle
The adiabatic approximation I

A classical example: a pendulum with \textit{slowly} varying length:

\[ l_s, \quad s = \epsilon t \]

Here \( t \) is the physical time, in the regime:

\[ \epsilon \ll 1, \quad t \sim \epsilon^{-1} \gg 1, \quad s \sim 1 \]

Two a priori different periods of oscillations:

\( \Delta T_\epsilon(s) \) associated with the \textit{driven dynamics} (Newton’s equations)

\( \Delta T_s \) associated with the \textit{instantaneous length}

Adiabatic approximation: during the driving,

\[ |T_\epsilon(s) - T_s| \leq C\epsilon, \quad s \in [0, 1] \]
Slow driven dynamics

Evolution equation in a Banach space $\mathcal{B}$

$$\dot{\varphi}(t) = L(\varphi(t), \alpha_t), \quad \varphi \in \mathcal{B},$$

$L : \mathcal{B} \times \mathbb{R} \to \mathcal{B}$ smooth
$t \mapsto \alpha_t$ given: parametric dependence describes the driving

Assumption. Convergence to fixpoint

$$\varphi(t) \to \varphi_{\alpha}, \quad L(\varphi_{\alpha}, \alpha) = 0,$$

for constant (frozen) $\alpha$.

Adiabatic Principle. If $\alpha$ changes slowly in time and if $\varphi(0) \sim \varphi_{\alpha_0}$ then

$$\varphi(t) \sim \varphi_{\alpha_t}$$

for all $t \in [0, T]$
In quantum mechanics

A smooth family $H_s = H_s^*$ of Hamiltonians on a Hilbert space $\mathcal{H}$

Spectrum with uniform gap $\gamma$

\[ \Sigma_{s,1} \quad \text{and} \quad \Sigma_{s,2} \]

and $P_s$ is the spectral projection associated with $\Sigma_{s,1}$

Dynamical equation (recall $s = \epsilon t$):

\[
\begin{align*}
    i\epsilon &\dot{\rho}_\epsilon(s) = [H_s, \rho_\epsilon(s)], \quad s \in [0, 1] \\
    \rho_\epsilon(0) &= P_0
\end{align*}
\]

The adiabatic theorem [Born, Fock, Kato,...]:

\[ \|\rho_\epsilon(s) - P_s\| \leq C\epsilon, \quad s \in [0, 1] \]
Proof I

We call

\[ L_s = -i [H_s, \cdot] \]

Equation for the error \( r(s) = \rho \epsilon(s) - P_s \):

\[
\left( \epsilon \frac{d}{ds} - L_s \right) r(s) = -\epsilon \dot{P}_s, \quad \text{since} \quad L_s P_s = 0
\]

By Duhamel’s principle

\[
r(s) = - \int_0^s \sigma^{s,s'}(\dot{P}_{s'}) ds'
\]

where \( \sigma^{s,s'} \) is the Heisenberg flow

\[
\left( \epsilon \frac{d}{ds} - L_s \right) \sigma^{s,s'}(O) = 0
\]
Proof II

Continue from

\[ \rho_\epsilon(s) = P_s - \int_0^s \sigma^{s,s'}(\dot{P}_{s'}) ds' \]

Using \( \sigma^{s,s'} \circ \sigma^{s',s} = \text{id} \)

\[ \epsilon \frac{d}{ds'} \sigma^{s,s'}(\zeta) = -\sigma^{s,s'}(L_{s'} \zeta) \]

to integrate by parts:

\[ \rho_\epsilon(s) - P_s = \epsilon \int_0^s \frac{d}{ds'} \left( \sigma^{s,s'}(L_{s'}^{-1} \dot{P}_{s'}) \right) ds' - \epsilon \int_0^s \sigma^{s,s'} \left( \frac{d}{ds'}(L_{s'}^{-1} \dot{P}_{s'}) \right) ds' \]
Observations

If $\dot{P}_0 = \dot{P}_1 = 0$

$$\rho_\epsilon(1) - P_1 = \epsilon \int_0^1 \sigma^{1,s} \left( \frac{d}{ds} (L_s^{-1} \dot{P}_s) \right) ds$$

Conclude by

$$\|\sigma^{1,s}(O)\| = \|O\| \quad \text{and} \quad \|L_s^{-1} \dot{P}_s\| < C$$

▷ The argument needs the uniform topology
▷ The key step (recall $L_s = -i[H_s, \cdot]$)

$$\dot{P}_s = L_sL_s^{-1} \dot{P}_s$$

presupposes that $\dot{P}_s \notin \text{Ker}(L_s)$ and a spectral gap
Inverting $L$

Spectrum of $L$:

$$L|\psi_i\rangle\langle\psi_j| = H|\psi_i\rangle\langle\psi_j| - |\psi_i\rangle\langle\psi_j|H = (E_i - E_j)|\psi_i\rangle\langle\psi_j|$$

Define

$$Q(A) = PA(1 - P) + (1 - P)AP$$

and note

$$Q(L(A)) = LQ(A)$$

If $\gamma$ is the spectral gap of $H$, then

$$L \upharpoonright \text{Ran} Q$$

has a gap in $(-\gamma, \gamma)$. Hence

$$\|L^{-1}Q(A)\| \leq C \frac{\|Q(A)\|}{\gamma}$$
Adiabatic estimate: the gapped case

\[ P^2 = P \] implies

\[ \dot{P} = Q(\dot{P}) \]

Riesz’ formula

\[ P = -\frac{1}{2\pi i} \int_{\Gamma} (H - z)^{-1} \, dz, \quad \dot{P} = -\frac{1}{2\pi i} \int_{\Gamma} (H - z)^{-1} \dot{H} (H - z)^{-1} \, dz \]

and the gap assumption yields

\[ \| \dot{P} \| \leq C \frac{\| \dot{H} \|}{\gamma} \]

Conclusion

\[ \| \rho_\epsilon(s) - P_s \| \leq C \epsilon \sup_{s \in [0,1]} \frac{\| \dot{H}_s \|}{\gamma_s^2} \]
Remarks

Other proved cases:

- In the gapless case, the error $O(\epsilon)$ replaced by $o(1)$, see J.E. Avron, A. Elgart, Commun. Math. Phys., 1999
- For non-linear evolution equations with small data, see Z. Gang, P. Grech, Commun. PDE, 2017

Missing:

- The statistical mechanical setting: quasi-static processes at positive temperature
- The quantum information setting: extended systems at zero temperature
A variational example

The **Gross-Pitaevskii** equation in $L^2(\mathbb{R}^3)$:

$$i\epsilon \dot{\Psi}_\epsilon(s) = -\Delta \Psi_\epsilon(s) + V_s \Psi_\epsilon(s) + |\Psi_\epsilon(s)|^2 \Psi_\epsilon(s)$$

A **ground state** $\Phi_s$ is a minimizer of the functional

$$\mathcal{E}_s(\Phi) = \int \left( \frac{1}{2} |\nabla \Phi|^2 + V_s |\Phi|^2 + \frac{1}{4} |\Phi|^4 \right)$$

with $\|\Phi\|_{L^2(\mathbb{R}^3)} = \eta$ fixed.

The **non-linear** adiabatic theorem: If $\Psi_\epsilon(0)$ is a ground state, then

$$\|\Psi_\epsilon(s) - \Phi_s\|_{H^2(\mathbb{R}^3)} \leq C \epsilon$$

Describes the adiabatic dynamics of $N$ bosons in the mean-field limit
Exponential estimates I

In the Schrödinger picture

\[ i\epsilon \dot{\psi}_\epsilon(s) = H_s \psi_\epsilon(s) \]

Isolated eigenvalue

\[ H_s \Omega_s = E_s \Omega_s \]

Adiabatic expansion Ansatz

\[ \psi_\epsilon(s) = e^{-\frac{i}{\epsilon} \int_0^s E_s \,dr} \left( \varphi^0(s) + \epsilon \varphi^1(s) + \epsilon^2 \varphi^2(s) + \cdots \right) \]

and solve order by order:

▷ Order 0: \( E \varphi^0 = H \varphi^0 \) yields \( \varphi^0 \) proportional to \( \Omega \):

\[ \varphi^0 = A^0 \Omega \]
Order 1: \( i \dot{\varphi}^0 + E \varphi^1 = H \varphi^1 \) yields

\[
i \dot{A}^0 \Omega + i A^0 \dot{\Omega} = (H - E) \varphi^1
\]

Decompose on \( \text{span}\{\Omega\} \oplus \text{span}\{\Omega\}^\perp \)

\[
\dot{A}^0 = 0
\]

\[
i A^0 \dot{\Omega} = (H - E) \varphi^1
\]

namely \( A_0 = 1 \) and

\[
\varphi^1 = A^1 \Omega + i(H - E)^{-1} \dot{\Omega}
\]

Notes:

- The ground state amplitude \( A^1 \) to be determined by the next order
- Well-defined resolvent since \( \dot{\Omega} \perp \Omega \)
Exponential estimates, conclusion

Repeat inductively to obtain

\[
\psi^N_\epsilon (s) = e^{-\frac{i}{\epsilon} \int_0^s E_s \, dr} \left( \varphi^0(s) + \cdots + \epsilon^N \varphi^N(s) + \epsilon^{N+1} \varphi^N_\perp(s) \right)
\]

▷ Key step: Decomposition \( \text{span}\{\Omega\} \oplus \text{span}\{\Omega\}^\perp \) with a gap \( \gamma \)
▷ Control up to order \( k \) requires \( H \in C^{k+1} \)
▷ If \( H \in C^\infty \), the error estimate \( O(\epsilon^\infty) \). In fact

\[
\| \psi_\epsilon - \psi^N_\epsilon \| \leq \kappa(\gamma) e^{-\frac{C(\gamma)}{\epsilon}}
\]

The many-body problem

Recall

$$\|\rho_\epsilon(s) - P_s\| \leq C(\gamma) \epsilon \sup_{s\in[0,1]} \|\dot{H}_s\|$$

The energy is extensive: In a volume $V$,

$$\|H_s\| \sim V$$

and if the driving is uniform

$$\|\dot{H}_s\| \sim V$$

The adiabatic estimate is meaningful only in the few-body regime:

$$\epsilon V \ll 1$$

What about the many-body regime $\epsilon \ll 1$ uniformly as $V \to \infty$?
Infrared catastrophe

A non-interacting spin-1/2 chain

\[ H_{[-L,L],h} = \sum_{x=-L}^{L} h \cdot \sigma_x \]

where \( \sigma_x \) is the vector of Pauli matrices at site \( x \), and \( h \in S^2 \)

\[ \Omega_0 = \cdots \otimes |0\rangle \otimes \cdots \otimes |0\rangle \otimes \cdots \]

\[ \Omega_\theta = \cdots \otimes |\theta\rangle \otimes \cdots \otimes |\theta\rangle \otimes \cdots \]

Many-body catastrophe: \( \Omega_\theta \perp \Omega_0 \) for large \( L \)

\[ \langle \Omega_\theta, \Omega_0 \rangle = (\cos \theta)^{2L+1} \rightarrow 0 \quad (L \rightarrow \infty) \]
PART II

Locality in quantum spin systems
Extended systems

Two kinds of many-body quantum systems

- Quantum spin systems (QSS): A countable collection of finite-dimensional quantum systems
- Interacting fermions hopping on a lattice

In both cases:

- The ‘lattice’ is equipped with a metric: notion of locality
- The infinite volume, finite density, limit can be controlled

Here, only QSS. For lattice fermions, see
B. Nachtergaele, R. Sims, A. Young, arXiv:1705.08553
and for adiabatic theorems:
D. Monaco, S. Teufel, arXiv:1707.01852
Quantum spin systems

A countable collection of quantum systems, labelled by \( x \in \Gamma \) with finite dimensional Hilbert spaces \( \mathcal{H}_x \)

Typically:

\( \Gamma = \mathbb{Z}^d \) equipped with graph distance \( d(\cdot, \cdot) \)
\( \mathcal{H}_x = \mathbb{C}^{2S+1} \) the Hilbert space of states of a spin-\( S \)

For finite \( \Lambda \subset \Gamma \)

\( \triangleright \) Local Hilbert space

\[ \mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{H}_x \]

\( \triangleright \) The algebra of local observables

\[ \mathcal{A}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{L}(\mathcal{H}_x) = \mathcal{L}(\mathcal{H}_\Lambda) \]
Quantum spin systems

Natural identification $\Lambda_1 \subset \Lambda_2 \Rightarrow \mathcal{A}_{\Lambda_1} \subset \mathcal{A}_{\Lambda_2}$ by

$$B \in \mathcal{A}_{\Lambda_1} \iff B \otimes 1_{\Lambda_2 \setminus \Lambda_1} \in \mathcal{A}_{\Lambda_2}$$

An interaction is a map

$$\Phi : Z \mapsto \Phi(Z) = \Phi(Z)^* \in \mathcal{A}_Z$$

and $\|\Phi(Z)\|$ decays in the diameter of $Z$

A Hamiltonian is an extensive operator

$$H_\Lambda = \sum_{Z \subset \Lambda} \Phi(Z) \in \mathcal{L}(\mathcal{H}_\Lambda)$$

Note: The decomposition of $H_\Lambda$ is not unique
Example

The toric code Hamiltonian

$\Lambda$: set of edges

$$H_{\Lambda, h} = - \sum_{v: \text{vertex}} A_v - \sum_{f: \text{face}} B_f - \sum_{e \in \Lambda} h \sigma^3_e$$

where

$$A_v = \prod_{i \in v} \sigma^1_i, \quad B_f = \prod_{i \in f} \sigma^3_i$$

Can be defined on $\mathbb{Z}^2$
or on cell decompositions of compact 2d surfaces

Topologically ordered ground states, see B. Nachtergaele’s talk
Dynamics and states: Heisenberg picture

Finite volume dynamics

\[ \tau_{t,\Lambda}(A) = e^{i t H_\Lambda} A e^{-i t H_\Lambda} \]

The operator \( H_\Lambda \) has no limit as \( \Lambda \to \Gamma \), but the dynamics does

\[ \tau^H_t(A) = \lim_{\Lambda \to \Gamma} \tau_{t,\Lambda}(A), \quad A \in \mathcal{L}(\mathcal{H}_Z) \]

States are a normalized positive linear functionals

\[ A \mapsto \text{Tr}(\rho_\Lambda A) \in \mathbb{C} \]

with dynamics

\[ i \frac{d}{dt} \rho_\Lambda(t) = [H_\Lambda, \rho_\Lambda(t)] \]
Locality I

Let $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$ with $d(X, Y) = d > 0$. Then

$$[\tau_{0,\Lambda}(A), B] = [A, B] = [A \otimes 1_{\Lambda \setminus X}, 1_{\Lambda \setminus Y} \otimes B] = 0.$$ 

Propagation estimate: the Lieb-Robinson bound (LRB)

$$\| [\tau_{t,\Lambda}(A), B] \| \leq C(A, B) e^{-\mu (d-v|t|)}$$

for constant $\mu, v > 0$.

- Message: For times $|t| \leq d/v$, $\tau_{t,\Lambda}(A)$ almost commutes with $B$
- Rate $\mu$ depends on the decay of $\Phi$
- Key to prove the existence of the limit $\tau_{t,\Lambda}(A)$ as $\Lambda \to \Gamma$

The Schrödinger equation has infinite propagation speed, but...

Corollary of the LRB: for any $\delta > 0$ and any $t \in \mathbb{R}$, there exists $A^\delta_t \in A_{X^{|v|t|+\delta}}$ such that

$$\|\tau_t(A) - A^\delta_t\| \leq C(A) e^{-\mu \delta}$$

where $X^r := \{x \in \mathbb{Z}^d : \text{dist}(x, X) \leq r\}$
Almost commuting operators

Let $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. If $A \in \mathcal{L}(\mathcal{H}_1)$ and $B \in \mathcal{L}(\mathcal{H}_2)$, then $[A, B] = 0$.

Also: if $A \in \mathcal{L}(\mathcal{H})$ and

$$[A, B] = 0 \quad \forall B \in \mathcal{L}(\mathcal{H}_2) \quad \implies \quad A \in \mathcal{L}(\mathcal{H}_1)$$

More generally: If $A \in \mathcal{L}(\mathcal{H})$ is such that

$$\|[A, 1 \otimes B]\| \leq \epsilon \|B\|$$

for all $B \in \mathcal{L}(\mathcal{H}_2)$, then there exists $\Pi(A) \in \mathcal{L}(\mathcal{H}_1)$ such that

$$\|\Pi(A) \otimes 1 - A\| \leq 2\epsilon.$$ 

In other words, almost commutation implies almost localization.
Almost commuting operators

Let

\[ \Pi(A) = \int_{\mathcal{U}(\mathcal{H}_2)} (1_{\mathcal{H}_1} \otimes U)^* A (1_{\mathcal{H}_1} \otimes U) d\mu(U) \]

where \( \mu \) is the normalized Haar measure \( \mu(S) = \mu(SU'), \ U' \in \mathcal{U}(\mathcal{H}_2) \)

Then

\[ \Delta \Pi(A) = \Pi(A) \otimes 1_{\mathcal{H}_2} \text{ because} \]

\[ \Pi(A)(1_{\mathcal{H}_1} \otimes V) = \int_{\mathcal{U}(\mathcal{H}_2)} (1_{\mathcal{H}_1} \otimes \tilde{U}V^*)^* A (1_{\mathcal{H}_1} \otimes \tilde{U}) d\mu(\tilde{U}V^*) \]

\[ = (1_{\mathcal{H}_1} \otimes V)\Pi(A) \]

\[ \Delta \text{ Approximation} \]

\[ \Pi(A) \otimes 1_{\mathcal{H}_2} - A = \int_{\mathcal{U}(\mathcal{H}_2)} (1_{\mathcal{H}_1} \otimes U)^* [A, (1_{\mathcal{H}_1} \otimes U)] d\mu(U) \]
Parallel transport

Back to spectral projections $P_s$ and vectors $P_s \Omega_s = \Omega_s$, namely

$$\dot{P}_s \Omega_s = (1 - P_s) \dot{\Omega}_s$$

Motion of $\Omega_s$ within $\text{Ran}(P_s)$ free.

A natural choice: No motion, a.k.a. parallel transport

$$P_s \dot{\Omega}_s = 0$$

yields

$$\dot{\Omega}_s = \dot{P}_s \Omega_s = (1 - P_s) \dot{P}_s P_s \Omega_s = [\dot{P}_s, P_s] \Omega_s$$

Kato's flow of ground states

$$\Omega_s = U^K(s) \Omega_0, \quad i\dot{U}(s) = i[\dot{P}_s, P_s] U^K(s)$$

Two choices made:

- Choice of flow within $\text{Ran}(P_s): \Omega_0 \mapsto \Omega_s$
- Choice of unitary to implement the flow $\Omega_s = U(s)\Omega_0$

But: the self-adjoint generator

$$G^K_s = i[\dot{P}_s, P_s]$$

is non-local.

Better (local) choice?

$$P_s = U(s)P_0U(s)^*$$

with generator

$$G_s = i\dot{U}(s)U(s)^* = -iU(s)\dot{U}(s)^*$$, namely

$$\dot{P}_s = -i[G_s, P_s]$$
Quasi-adiabatic flow I

Yes, using the gap! Hastings’ generator:

\[ G_s^H = \int_{\mathbb{R}} W(t) e^{itH_s} \dot{H}_s e^{-itH_s} dt \]

where \( W \in L^1(\mathbb{R}; \mathbb{R}) \)


▷ Good flow: \([H, P] = 0\) implies \([\dot{H}, P] = [\dot{P}, H]\), hence

\[-i[G^H, P] = -i \int_{\mathbb{R}} W(t) e^{itH} [\dot{P}, H] e^{-itH} dt = \int_{\mathbb{R}} W(t) \frac{d}{dt} \tau_t(\dot{P}) dt\]

Choice 1: \( W(t) = \Theta(t) - 1/2 \) yields \( \dot{P} \) but divergent
Quasi-adiabatic flow II

\[-i [G^H, P] = \int_{\mathbb{R}} W(t) \frac{d}{dt} \tau_t(\dot{P}) dt\]

Choice 2: \(W(t) = \Theta(t) - 1/2 + T(t)\) with

\[\hat{T}'(\xi) = 0 \quad \text{whenever} \quad |\xi| \geq \gamma.\]

because \((e^{-iHt} = \int e^{-i\lambda t} dE(\lambda))\)

\[-\int T'(t)e^{iHt} \dot{P}e^{-iHt} dt = -\sqrt{2\pi} \int \hat{T}'(\mu - \lambda)dE(\lambda) \dot{P}dE(\mu) = 0\]

since \(\dot{P}\) is off-diagonal

▷ Parallel transport: If \(\hat{W}(0) = 0\) ie \(W\) is odd, then

\[PG^H P = \int W(t) P \dot{H} P dt = \sqrt{2\pi} \hat{W}(0) P \dot{H} P = 0\]
Quasi-adiabatic flow: Locality

Finally assume fast decay

\[ W(t) = \mathcal{O}(|t|^{-\infty}) \]

Then for \( A \in \mathcal{A}_X \)

\[ \int W(t) \tau_t(A) dt \]

is almost local:

- For \( |t| \leq T \), \( \tau_t(A) \) is supported near \( X \)
- For \( |t| \geq T \), use the decay of \( W \)

Hence

\[ G^H = \int W(t) \tau_t(\dot{H}) dt = \sum Z \int W(t) \tau_t(\dot{\Phi}(Z)) dt \]

is a sum of almost local terms, a.k.a. a Hamiltonian
Summary and perspective

- Both $G^K, G^H$ generate parallel transport
- $G^K$ has no good locality properties
- $G^H$ generates $U^H$ satisfying a Lieb-Robinson bound: suited for many-body applications
- So far: no adiabatic dynamics (no $\epsilon$), but anticipating:

  $U^H_s$ is the zeroth order of the local adiabatic expansion of the Schrödinger unitary $U_\epsilon(s)$
PART III

Many-body adiabatic theorem
Recap: the goal

Time dependent quantum spin Hamiltonian in volume $\Lambda$

$$H_{\Lambda,s} = \sum_{Z \subset \Lambda} \Phi_s(Z)$$

with gapped spectral projection $P_{\Lambda,s}$

Dynamics

$$\begin{cases} 
  i\epsilon \dot{\psi}_{\Lambda,\epsilon}(s) = H_{\Lambda,s}\psi_{\Lambda,\epsilon}(s), & s \in [0, 1] \\
  \psi_{\Lambda,\epsilon}(0) = \Omega_{\Lambda,0} 
\end{cases}$$

where $\Omega_{\Lambda,0} = P_{\Lambda,0}\Omega_{\Lambda,0}$

Goal: control the long time dynamics

$$\psi_{\Lambda,\epsilon}(s) - \Omega_{\Lambda,s} = O(\epsilon), \quad s \in [0, 1]$$

uniformly in the volume. But in which sense?
Local changes

Uniform topology is too strong (infrared catastrophe)
Test $\psi_{\Lambda,\epsilon}(s)$ only locally:

$$\langle \psi_{\Lambda,\epsilon}(s), O\psi_{\Lambda,\epsilon}(s) \rangle - \langle \tilde{\psi}_{\Lambda,\epsilon}(s), O\tilde{\psi}_{\Lambda,\epsilon}(s) \rangle = O(\epsilon)$$

where

$$\tilde{\psi}_{\epsilon}(s) = P_s \tilde{\psi}_{\epsilon}(s)$$

and $O$ is local, arbitrary but fixed
No uniformity in $O$, expect error bounds to depend on

$$\|O\|, \quad |\text{supp}(O)|,$$

but not on $\Lambda$
Many-body adiabatic theorem

Spectral gap: With \( E_{\Lambda,s} := \inf \text{Spec}(H_{\Lambda,s}) \),

\[
\gamma := \inf_{\Lambda \in \mathcal{F}(\Gamma), s \in [0,1]} \sup \{ \delta : (E_{\Lambda,s}, E_{\Lambda,s} + \delta) \cap \text{Spec}(H_{\Lambda,s}) = \emptyset \}
\]

Theorem [SB-Fraas-De Roeck]
Let \( \Gamma \) be \( d \)-dimensional. If the Hamiltonian is

\( \triangleright \) gapped: \( \gamma > 0 \),
\( \triangleright \) smooth: \( H_s \in C^{d+1}([0,1]) \)

then there is \( \tilde{\psi}_{\Lambda,\epsilon}(s) = P_{\Lambda,s}\tilde{\psi}_{\Lambda,\epsilon}(s) \) such that

\[
\left| \langle \psi_{\Lambda,\epsilon}(s), O\psi_{\Lambda,\epsilon}(s) \rangle - \langle \tilde{\psi}_{\Lambda,\epsilon}(s), O\tilde{\psi}_{\Lambda,\epsilon}(s) \rangle \right| \leq C|\text{supp}(O)|^2\|O\|\epsilon
\]

for all \( \Lambda \in \mathcal{F}(\Gamma) \), with \( C \) independent of \( \Lambda \)
On parallel transport

If \( \text{Ran}(P_{\Lambda,s}) \) is nearly degenerate of width

\[ w \leq C \min \left\{ \epsilon^2, \frac{\epsilon}{|\Lambda|} \right\} \]

Then

\[ \tilde{\psi}_{\Lambda,\epsilon}(s) = \Omega_{\Lambda}(s) \]

is parallel transported

\[ i\dot{\Omega}_{\Lambda}(s) = i[\dot{P}_{\Lambda,s}, P_{\Lambda,s}]\Omega_{\Lambda}(s) \]

and independent of \( \epsilon \)
Locality again

Looking at expectation values of local observables has

- An advantage: Essentially ignores $\Lambda$
- A disadvantage: Recall

$$\rho_\epsilon(s) - P_s = \epsilon \int_0^s \sigma^{s,s'} \left( \frac{d}{ds'} (L_{s'}^{-1} \dot{P}_{s'}) \right) ds'$$

$\sigma^{s,s'}$ is norm preserving

but it spreads local observables over volume

$$|s' - s|^d = \epsilon^{-d} |t' - t|^d$$

yielding a naive bound

$$O(\epsilon^{1-d}) \quad \text{as} \quad \epsilon \to 0$$

Solution: Expand to higher order in adiabatic perturbation theory
The map $\mathcal{I}$

Define $\mathcal{I}_s : \mathcal{A} \to \mathcal{A}$ by

$$\mathcal{I}_s(A) := \int_{\mathbb{R}} W(t)e^{itH_s}Ae^{-itH_s}dt$$

where $W \in L^1(\mathbb{R})$ decays faster than any inverse power and such that

$$\hat{W}(\xi) = \frac{-i}{\sqrt{2\pi\xi}}, \quad \text{if} \quad |\xi| \geq \gamma.$$ 

Fact: If $A$ is local, then $\mathcal{I}_s(A)$ is almost local

- for short times: $\tau_t(A)$ almost local by LRB
- for long times: $W(t)\|\tau_t(A)\| = W(t)\|A\|$ vanishes by decay of $W$
Local spectral flow

Claim: $\mathcal{I}(\cdot)$ is a local inverse of $-i[H, \cdot]$

$$A = -i[H, \mathcal{I}(A)]$$

whenever

$$A = PA(1 - P) + (1 - P)AP$$

Indeed

$$\mathcal{I}(A) = \int W(t)e^{i\lambda t}e^{-i\mu t}dE_{\lambda}AdE_{\mu}dt = \sqrt{2\pi} \int \hat{W}(\mu - \lambda)dE_{\lambda}AdE_{\mu}$$

and

$$-i[H, \mathcal{I}(A)] = -i\sqrt{2\pi} \int_{|\lambda - \mu| \geq \gamma} (\lambda - \mu)\hat{W}(\mu - \lambda)dE_{\lambda}AdE_{\mu}$$

$$= -i\sqrt{2\pi} \int_{|\lambda - \mu| \geq \gamma} (\lambda - \mu) \frac{i}{\sqrt{2\pi(\lambda - \mu)}}dE_{\lambda}AdE_{\mu} = A$$
Quasi-adiabatic evolution, revisited

For a spectral projection

\[ \dot{P}_s = P_s \dot{P}_s (1 - P_s) + (1 - P_s) \dot{P}_s P_s \]

Hence

\[ \dot{P}_s = -i[H_s, \mathcal{I}_s(\dot{P}_s)] = -i\mathcal{I}_s([H_s, \dot{P}_s]) = i\mathcal{I}_s([\dot{H}_s, P_s]) \]

denamely

\[ \dot{P}_s = i[\mathcal{I}_s(\dot{H}_s), P_s] \]

and

\[ G^H_s = \mathcal{I}_s(\dot{H}_s) = \sum Z \mathcal{I}_s(\dot{\Phi}_s(Z)) \]

is a sum of almost local terms
Local spectral flow

Another consequence: For any $B$, $[B, P]$ is off-diagonal, hence

$$[B, P] = -i[H, \mathcal{I}([B, P])]$$
$$= -i[H, [\mathcal{I}(B), P]]$$
$$= -i[[H, \mathcal{I}(B)], P]$$

by Jacobi’s identity and

$$[H, P] = 0$$
The dressed states

Key idea:
Compare the evolved $\rho_\epsilon(s)$ with a dressed ground state projection

$$\Pi_\epsilon(s) = V_\epsilon(s) P_s V_\epsilon(s)^*, \quad V_\epsilon(s) = e^{i\epsilon A_1(s) + \epsilon^2 A_2(s) + \cdots}$$

such that $A_j$'s are extensive operators, $\Pi_\epsilon(s)$ solves

$$i\epsilon \dot{\Pi}_\epsilon(s) = [H_s + R_\epsilon(s), \Pi_\epsilon(s)]$$

where $R_\epsilon(s)$ is $O(\epsilon^n)$

Then, when tested against a local observable:

$\triangleright$ $\rho_\epsilon(s) - \Pi_\epsilon(s)$ is of order $\epsilon^{n-d}$ since

$$i\epsilon \dot{\rho}_\epsilon = [H_s, \rho_\epsilon(s)]$$

$\triangleright$ $\Pi_\epsilon(s) - P_s$ is of order $\epsilon$
Adiabatic expansion I

Ansatz $\Pi = V \cdot PV^*$

$$i\epsilon \dot{\Pi} = [H, \Pi] + V \left[ i\epsilon V^* \dot{V} - \epsilon \mathcal{I}(\dot{H}) + (H - V^* HV), P \right] V^*$$

with

$$V = e^{i\epsilon A_1 + \epsilon^2 A_2 + \cdots}$$

Now

$$H - V^* HV = i\epsilon [A_1, H] + \mathcal{O}(\epsilon^2)$$

and

$$\dot{V} = \mathcal{O}(\epsilon)$$

No zeroth order: The zeroth order dynamics is fully within $\text{Ran} P$
Adiabatic expansion II

First order is zero if

\[
i[[A_1, H], P] = [\mathcal{I}(\dot{H}), P]
\]

to be solved for \( A_1 \). By the local inverse lemma

\[
A_1 = \mathcal{I}\left(\mathcal{I}(\dot{H})\right)
\]

Continue inductively: By choosing \( A_1, \ldots A_n \), the remainder

\[
V^* RV = i\epsilon V^* \dot{V} - \epsilon \mathcal{I}(\dot{H}) + H - V^* HV
\]

is of order \( \epsilon^{n+1} \).

In which sense? All \( A_j \)'s are local extensive operators, so that

\[
|\text{Tr}(V^* RVO)| \leq C\|O\|\|\text{supp}(O)\|\epsilon^{n+1}
\]
Proof I

Starting from a $\Omega_0 \in \text{Ran}P_0$, define

$$\phi_{n, \epsilon}(s) = V_{n, \epsilon}(s, 0)\Omega_0$$

where

$$i\epsilon \dot{V}_{n, \epsilon}(s, s') = (H_s + R_{n, \epsilon})V_{n, \epsilon}(s, s'), \quad V_{n, \epsilon}(s', s') = 1$$

Then

$$\langle \psi_{\epsilon}(s), O\psi_{\epsilon}(s) \rangle - \langle \phi_{n, \epsilon}(s), O\phi_{n, \epsilon}(s) \rangle = -\frac{i}{\epsilon} \int_0^s \langle \phi_{n, \epsilon}(r), [R_{n, \epsilon}(r), U_{\epsilon}(s, r)^*OU_{\epsilon}(s, r)] \phi_{n, \epsilon}(r) \rangle dr$$

$$= O(\epsilon^{-1+(n+1)-d})$$
Proof II

By construction

\[ \phi_{n,\epsilon}(s) \in \text{Ran}(\Pi_{n,\epsilon}(s)) \]

so that there is \( \tilde{\psi}_{n,\epsilon}(s) \in \text{Ran} P_s \) such that

\[ \langle \tilde{\psi}_{n,\epsilon}(s), O\tilde{\psi}_{n,\epsilon}(s) \rangle - \langle \phi_{n,\epsilon}(s), O\phi_{n,\epsilon}(s) \rangle = O(\epsilon) \]

Summarizing:

- \( \psi_\epsilon(s) \) the dynamically evolved vector
- \( \phi_{n,\epsilon}(s) \) a ‘dressed’ state
- \( \tilde{\psi}_{n,\epsilon}(s) \) a state in \( \text{Ran} P_s \)

with

\[ \psi_\epsilon(s) = \phi_{n,\epsilon}(s) + O(\epsilon^{n-d}), \quad \phi_{n,\epsilon}(s) = \tilde{\psi}_{n,\epsilon}(s) + O(\epsilon) \]

Choose \( n \geq d + 1 \)
Remarks

- Differentiability assumption
  \[ A_1(s) = \mathcal{I}_s(\mathcal{I}_s(\dot{H}_s)) \]

  Generally: \( A_n(s) \) depends on \( H_s^{(k)} \), for \( 1 \leq k \leq n \)

- Local-in-time dependence: If \( H_{s=1}^{(k)} = 0 \) for \( 1 \leq k \leq n \), then
  \[ \Pi_{n,\epsilon}(1) - P_1 = O(\epsilon^n) \]
  yielding an improved adiabatic estimate: e.g. \( O(\epsilon^\infty) \) if \( H \in C^\infty \)

- Parallel transport: The dynamics of \( \tilde{\psi}_{n,\epsilon}(s) \) is generated by
  \[ i[A_1, H]P = i[[A_1, H], P]P = i[\mathcal{I}(\dot{H}), P]P = i[G^H, P]P \]
  to \( O(\epsilon) \)
On the gap assumption

Crucial assumption: open spectral gap, uniformly in $\Lambda, s$

- When can it be proved?
  Perturbations of frustration-free systems

  Perturbations of non-interacting fermion systems
  M.B. Hasting, arXiv:1706.02270
  B. Nachtergaele, R. Sims, A. Young, arXiv:1705.08553

- What about gapless (disordered) systems?
PART IV

Linear response theory
Motivation

Ohm’s law of electric conductivity, Fourier’s law of thermal conductivity

General form

\[ J = \chi F \]

\( J \) is a current
\( F \) is a force
\( \chi \) is the matrix of linear response coefficients

Three connected mathematical problems:

- Validity of the linear relation
- Derivation of a general formula for \( \chi \)
- Computation of \( \chi \) in a given setting

One famous example (see A. Bols’ talk):
The quantisation of the quantum Hall conductivity
State-of-the-art

In Physics: Well understood

- Derivation by the first order perturbation theory
- Kubo formula (1957)
- Some doubts about the order of limits, van Kampen (1971)

In Math: Very few results

- One of the open problems on Simon’s 1984 list
- Non-interacting gapped systems (Elgart, Schlein 2004)
- Disordered systems (Bouclet et al 2005)
- Smoothed in frequency (Klein et al 2007, Bru, Pedra 2017)
- Compactly supported driving (Jakšić et al 2006)
Linear response: Setting

Setting: An adiabatically switched on perturbation

\[ H = H_i + e^{\alpha V}, \quad t \in (-\infty, 0] \]

with \( \alpha \in \mathbb{R} \). Also,

\[ P_i = P\{\alpha = 0, s\} = P\{\alpha, t=-\infty\} \]

the ground state projection of \( H_i \)

Linear response coefficient of a local observable \( J \):

\[ \chi_{J,V} := \lim_{\alpha \to 0} \lim_{\epsilon \to 0} \lim_{\Lambda \to \Gamma} \frac{\text{Tr}(J\rho_{\alpha,\epsilon}(0)) - \text{Tr}(JP_i)}{\alpha} \]

Example: \( V \) is an electromotive force
\( J \) is an electric current
\( \chi_{J,V} \) is the conductivity
Formal derivation I

Initial condition

\[ \lim_{t \to -\infty} e^{iH_i t} \rho_{\epsilon, \alpha}(t) e^{-iH_i t} = P_i \]

For \( \varrho(t) := e^{iH_i t} \rho_{\epsilon, \alpha}(t) e^{-iH_i t} \),

\[ \varrho_{\epsilon, \alpha}(t) = \varrho_{\epsilon, \alpha}(t_0) - i\alpha \int_{t_0}^{t} e^{\epsilon \tau} \left[ e^{iH_i \tau} V e^{-iH_i \tau}, \varrho_{\epsilon, \alpha}(\tau) \right] d\tau \]

Let \( t = 0, t_0 \to -\infty \),

\[ \rho_{\epsilon, \alpha}(0) - P_i = -i\alpha \int_{-\infty}^{0} e^{\epsilon \tau} e^{iH_i \tau} \left[ V, \rho_{\epsilon, \alpha}(\tau) \right] e^{-iH_i \tau} d\tau \]

to obtain to first order in \( \alpha \) (only linear response)

\[ \text{Tr} \left( J(\rho_{\epsilon, \alpha}(0) - P_i) \right) = -i\alpha \int_{-\infty}^{0} e^{\epsilon \tau} \text{Tr} \left( P_i \left[ e^{-iH_i \tau} J e^{iH_i \tau}, V \right] \right) d\tau \]
Formal derivation II

We obtained Kubo’s formula

\[ \chi_{J,V}^{\text{Kubo}} = \lim_{\epsilon \to 0} i \int_{-\infty}^{0} e^{\epsilon \tau} \text{Tr} \left( P_i \left[ V, e^{-iH_i \tau} J e^{iH_i \tau} \right] \right) d\tau \]

The catch:

\[ \| [V, e^{-iH_i \tau} J e^{iH_i \tau}] \| \sim \tau^d \]

and since

\[ \int e^{\epsilon \tau} \tau^d d\tau \sim \frac{1}{\epsilon^{d+1}} \]

the limit \( \epsilon \to 0 \) may not exist
By the adiabatic theorem

$$\rho_{\alpha, \epsilon}(0) - P_{\{\alpha, s=0\}} \to 0$$

locally as $\epsilon \to 0$, uniformly in $\Lambda$

Since $H_\alpha = H_i + \alpha V$,

$$\partial_\alpha H_\alpha = V$$

so that the spectral flow yields

$$\lim_{\alpha \to 0} \alpha^{-1} (P_{\{\alpha, s=0\}} - P_i) = i[I_i(V), P_i]$$

uniformly in $\Lambda$
The linear response coefficient

Hence \((\overline{\text{Tr}} = \sup_{\Lambda} \text{Tr})\)

\[
\alpha^{-1} \overline{\text{Tr}} \left( J(\rho_{\alpha,\epsilon}(0) - P_i) \right) \\
\leq \alpha^{-1} \overline{\text{Tr}} \left( J(\rho_{\alpha,\epsilon}(0) - P_{\{\alpha,s=0\}}) \right) + \alpha^{-1} \overline{\text{Tr}} \left( J(P_{\{\alpha,s=0\}} - P_i) \right) \\
\rightarrow \alpha^{-1} \overline{\text{Tr}} \left( J(P_{\{\alpha,s=0\}} - P_i) \right) \quad (\epsilon \rightarrow 0) \\
\rightarrow i\overline{\text{Tr}} \left( J[\mathcal{I}_i(V), P_i] \right) \quad (\alpha \rightarrow 0)
\]

**Theorem.** [SB-Fraas-de Roeck]

*If \(H = H_i + \beta V\) is gapped for in a neighbourhood of \(\beta = 0\), then*

\[
\chi_{J,V} = -i\text{Tr} \left( P_i[\mathcal{I}_i(V), J] \right)
\]

*And again: \(\chi_{J,V}\) expressed only in terms of ‘unperturbed’ quantities*
Kubo’s formula, again:

\[ \chi_{J,V}^{\text{Kubo}} = \lim_{\epsilon \to 0} i \int_{-\infty}^{0} e^{\epsilon \tau} \text{Tr} \left( P \left[ V, e^{-iH\tau} Je^{iH\tau} \right] \right) d\tau \]

Replace \( V \to \tilde{V} = PV(1 - P) + (1 - P)VP \) and use \( \tilde{V} = -i[H, \mathcal{I}(\tilde{V})] \)

Note

\[ i \int_{0}^{\infty} e^{-\epsilon \tau} \tau \left( [H, \mathcal{I}(\tilde{V})] \right) d\tau = \int \frac{(\mu - \lambda)}{(\mu - \lambda) - i\epsilon} \ dE(\lambda) \mathcal{I}(\tilde{V})dE(\mu) \rightarrow \mathcal{I}(\tilde{V}) \]

as \( \epsilon \to 0 \), because there is a gap and \( \mathcal{I}(\tilde{V}) \) is off-diagonal. Hence

\[ \chi_{J,V}^{\text{Kubo}} = -i \text{Tr} \left( P \left[ \mathcal{I}(V), J \right] \right) = \chi_{J,V} \]
Quantum Hall system: Two-parameter family

\[ H(\phi_1, \phi_2), \quad \phi_1, \phi_2 \in [0, 2\pi) \]

corresponding to an EMF in the 1 direction driving a current in the 2 direction

The electric potential:

\[ V = Q_1 \]

minimally coupled:

\[ -i[H, V] = \partial_{\phi_1} H \]

The local current:

\[ J = \partial_{\phi_2} H \]
Since $W$ is an odd function

\[ \chi_{J,V} = -i \text{Tr}(P[I(\tilde{V}), J]) = i \text{Tr}(P[\tilde{V}, I(J)]) \]

Using again $\tilde{V} = -i[H, I(\tilde{V})] = -iI([H, \tilde{V}])$,

\[ \chi_{J,V} = i \text{Tr}(P[I(\partial_{\phi_1} H), I(\partial_{\phi_2} H)] = i \text{Tr}(P[G^H_1, G^H_2]) \]

and since $P I(\partial_{\phi_j} H) P = 0$,

\[ \chi_{J,V} = i \text{Tr}(P [[I(\partial_{\phi_1} H), P], [I(\partial_{\phi_2} H), P]]) = i \text{Tr}(P[\partial_{\phi_1} P, \partial_{\phi_2} P]) \]

which is the **adiabatic curvature**, see J. Avron, R. Seiler and B. Simon, Phys. Rev. Lett., 1983
Summary

Assumption: Spectral gap

- Infrared catastrophe: Norm estimates fail for extended systems
- Change topology: consider only local expectation values
- Both Schrödinger dynamics and Hastings’ parallel transport are local
- Propagation still yields error $\mathcal{O}(\epsilon^{-d})$
- Solution: Go to higher order in adiabatic perturbation theory to get a many-body adiabatic theorem
- Crucially, all estimates are independent of the volume: derive validity of linear response theory

Another application of Hastings’ local flow:
Quantisation of conductance in many-body systems, see
SB, A. Bols, W. De Roeck and M. Fraas, Annales H. Poincaré, 2018