Many-body adiabatic theory

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‘Would you like to know what fascinates me in science? I find the ultimate poetry in it: in mathematics, the exhilarating vertigo of numbers; with astronomy, the mysterious whisper of the universe. But please, don’t mention truth!’

Amin Maalouf

$\alpha\delta\iota\beta\alpha\tau o\varsigma$, impassable

Plan of the course:

- ▷ Part 1. The adiabatic principle
- ▷ Part 2. Locality in quantum spin systems
- ▷ Part 3. The many-body adiabatic theorem
- ▷ Part 4. Linear response theory

References:

- ▷ *Elementary Exponential Error Estimates for the Adiabatic Approximation*

  G. Hagedorn and A. Joye, J. Math. Anal. Appl., 2002
- ▷ *Adiabatic Theorem for Quantum Spin Systems*

  SB, W. De Roeck, and M. Fraas, Phys. Rev. Lett. 2017
- ▷ *The adiabatic theorem and linear response theory for extended quantum systems*

  SB, W. De Roeck, and M. Fraas, Commun. Math. Phys. 2018

Many-body quantum systems

The general themes of the course

- Use locality properties, a.k.a. propagation estimates, of quantum systems to go beyond the noninteracting approximation in condensed matter physics
- Combine the long time limit and the thermodynamic limit for non-autonomous dynamics

Common issues to many current topics in quantum information theory, quantum control and quantum statistical mechanics

PART I

Adiabatic Principle

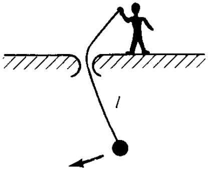
The adiabatic approximation I

A classical example: a pendulum with slowly varying length:

$$l_s, \quad s = \epsilon t$$

Here t is the physical time, in the regime:

$$\epsilon \ll 1, \qquad t \sim \epsilon^{-1} \gg 1, \qquad s \sim 1$$



Two a priori different periods of oscillations:

- $T_\epsilon(s)$ associated with the driven dynamics (Newton's equations)
- T_s associated with the instantaneous length

Adiabatic approximation: during the driving,

$$|T_\epsilon(s) - T_s| \leq C\epsilon, \quad s \in [0, 1]$$

Slow driven dynamics

Evolution equation in a Banach space $\mathcal{B}$

$$\dot{\varphi}(t) = L(\varphi(t), \alpha_t), \qquad \varphi \in \mathcal{B},$$

$L : \mathcal{B} \times \mathbb{R} \to \mathcal{B}$ smooth
$t \mapsto \alpha_t$ given: parametric dependence describes the driving

Assumption. Convergence to fixpoint

$$\varphi(t) \longrightarrow \varphi_\alpha, \qquad L(\varphi_\alpha, \alpha) = 0,$$

for constant (frozen) α.

Adiabatic Principle. If α changes slowly in time and if $\varphi(0) \sim \varphi_{\alpha_0}$ then

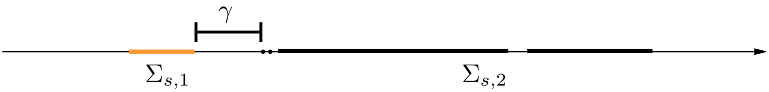
$$\varphi(t) \sim \varphi_{\alpha_t}$$

for all $t \in [0, T]$

In quantum mechanics

A smooth family $H_s = H_s^*$ of Hamiltonians on a Hilbert space $\mathcal{H}$

Spectrum with uniform gap γ



and P_s is the spectral projection associated with $\Sigma_{s,1}$

Dynamical equation (recall $s = \epsilon t$):

$$\begin{cases} \mathrm{i}\epsilon\dot{\rho}_\epsilon(s) = [H_s, \rho_\epsilon(s)], & s \in [0,1] \\ \rho_\epsilon(0) = P_0 \end{cases}$$

The adiabatic theorem [Born, Fock, Kato,...]:

$$\|\rho_\epsilon(s) - P_s\| \leq C\epsilon, \quad s \in [0,1]$$

Proof I

We call

$$L_s = -\mathrm{i}[H_s, \cdot]$$

Equation for the error $r(s) = \rho_\epsilon(s) - P_s$:

$$\left(\epsilon\frac{\mathrm{d}}{\mathrm{d}s} - L_s\right) r(s) = -\epsilon \dot{P}_s, \qquad \text{since } L_s P_s = 0$$

By Duhamel's principle

$$r(s) = -\int_0^s \sigma^{s,s'}(\dot{P}_{s'})\mathrm{d}s'$$

where $\sigma^{s,s'}$ is the Heisenberg flow

$$\left(\epsilon\frac{\mathrm{d}}{\mathrm{d}s} - L_s\right) \sigma^{s,s'}(O) = 0$$

Proof II

Continue from

$$\rho_\epsilon(s) = P_s - \int_0^s \sigma^{s,s'}(\dot{P}_{s'})\mathrm{d}s'$$

Using $\sigma^{s,s'} \circ \sigma^{s',s} = \mathrm{id}$

$$\epsilon \frac{\mathrm{d}}{\mathrm{d}s'}\sigma^{s,s'}(\zeta) = -\sigma^{s,s'}(L_{s'}\zeta)$$

to integrate by parts:

$$\rho_\epsilon(s) - P_s = \epsilon \int_0^s \frac{\mathrm{d}}{\mathrm{d}s'} \left(\sigma^{s,s'}(L_{s'}^{-1}\dot{P}_{s'})\right) \mathrm{d}s' - \epsilon \int_0^s \sigma^{s,s'} \left(\frac{\mathrm{d}}{\mathrm{d}s'}(L_{s'}^{-1}\dot{P}_{s'})\right) \mathrm{d}s'$$

Observations

If $\dot{P}_0 = \dot{P}_1 = 0$

$$\rho_\epsilon(1) - P_1 = \epsilon \int_0^1 \sigma^{1,s}\left(\frac{\mathrm{d}}{\mathrm{d}s}(L_s^{-1}\dot{P}_s)\right)\mathrm{d}s$$

Conclude by

$$\|\sigma^{1,s}(O)\| = \|O\| \quad \text{and} \quad \|L_s^{-1}\dot{P}_s\| < C$$

- ▷ The argument needs the uniform topology
- ▷ The key step (recall $L_s = -\mathrm{i}[H_s, \cdot]$)

$$\dot{P}_s = L_s L_s^{-1}\dot{P}_s$$

presupposes that $\dot{P}_s \notin \mathrm{Ker}(L_s)$ and a spectral gap

Inverting L

Spectrum of L:

$$L|\psi_i\rangle\langle\psi_j| = H|\psi_i\rangle\langle\psi_j| - |\psi_i\rangle\langle\psi_j|H = (E_i - E_j)|\psi_i\rangle\langle\psi_j|$$

Define

$$\mathcal{Q}(A) = PA(1-P) + (1-P)AP$$

and note

$$\mathcal{Q}(L(A)) = L\mathcal{Q}(A)$$

If γ is the spectral gap of H, then

$$L \restriction \mathrm{Ran}\mathcal{Q}$$

has a gap in $(-\gamma, \gamma)$. Hence

$$\|L^{-1}\mathcal{Q}(A)\| \leq C\frac{\|\mathcal{Q}(A)\|}{\gamma}$$

Adiabatic estimate: the gapped case

$P^2 = P$ implies

$$\dot{P} = \mathcal{Q}(\dot{P})$$

Riesz' formula

$$P = -\frac{1}{2\pi\mathrm{i}}\oint_\Gamma (H-z)^{-1}\mathrm{d}z, \qquad \dot{P} = -\frac{1}{2\pi\mathrm{i}}\oint_\Gamma (H-z)^{-1}\dot{H}(H-z)^{-1}\mathrm{d}z$$

and the gap assumption yields

$$\|\dot{P}\| \leq C\frac{\|\dot{H}\|}{\gamma}$$

Conclusion

$$\|\rho_\epsilon(s) - P_s\| \leq C\epsilon \sup_{s\in[0,1]} \frac{\|\dot{H}_s\|}{\gamma_s^2}$$

Remarks

Other proved cases:

- In the gapless case, the error $\mathcal{O}(\epsilon)$ replaced by $o(1)$, see
  J.E. Avron, A. Elgart, Commun. Math. Phys., 1999
- For non-linear evolution equations with small data, see
  Z. Gang, P. Grech, Commun. PDE, 2017
- Open quantum system, i.e. Lindblad evolutions, see
  J.E. Avron, M. Fraas, G.M. Graf, P. Grech, Commun. Math. Phys., 2012

Missing:

- The statistical mechanical setting: quasi-static processes at positive temperature
- The quantum information setting: extended systems at zero temperature

A variational example

The Gross-Pitaevskii equation in $L^2(\mathbb{R}^3)$:

$$\mathrm{i}\epsilon\dot{\Psi}_\epsilon(s) = -\Delta\Psi_\epsilon(s) + V_s\Psi_\epsilon(s) + |\Psi_\epsilon(s)|^2\Psi_\epsilon(s)$$

A ground state Φ_s is a minimizer of the functional

$$\mathcal{E}_s(\Phi) = \int \left(\frac{1}{2}|\nabla\Phi|^2 + V_s|\Phi|^2 + \frac{1}{4}|\Phi|^4\right)$$

with $\|\Phi\|_{L^2(\mathbb{R}^3)} = \eta$ fixed.

The non-linear adiabatic theorem: If $\Psi_\epsilon(0)$ is a ground state, then

$$\|\Psi_\epsilon(s) - \Phi_s\|_{H^2(\mathbb{R}^3)} \leq C\epsilon$$

Describes the adiabatic dynamics of N bosons in the mean-field limit

Exponential estimates I

In the Schrödinger picture

$$\mathrm{i}\epsilon\dot{\psi}_\epsilon(s) = H_s\psi_\epsilon(s)$$

Isolated eigenvalue

$$H_s\Omega_s = E_s\Omega_s$$

Adiabatic expansion Ansatz

$$\psi_\epsilon(s) = \mathrm{e}^{-\frac{\mathrm{i}}{\epsilon}\int_0^s E_s \mathrm{d}r}\left(\varphi^0(s) + \epsilon\varphi^1(s) + \epsilon^2\varphi^2(s) + \cdots\right)$$

and solve order by order:

- Order 0: $E\varphi^0 = H\varphi^0$ yields φ^0 proportional to Ω:

$$\varphi^0 = A^0\Omega$$

Exponential estimates II

- Order 1: $\mathrm{i}\dot{\varphi}^0 + E\varphi^1 = H\varphi^1$ yields

$$\mathrm{i}\dot{A}^0\Omega + \mathrm{i}A^0\dot{\Omega} = (H-E)\varphi^1$$

Decompose on $\mathrm{span}\{\Omega\} \oplus \mathrm{span}\{\Omega\}^\perp$

$$\dot{A}^0 = 0$$
$$\mathrm{i}A^0\dot{\Omega} = (H-E)\varphi^1$$

namely $A_0 = 1$ and

$$\varphi^1 = A^1\Omega + \mathrm{i}(H-E)^{-1}\dot{\Omega}$$

Notes:

- The ground state amplitude A^1 to be determined by the next order
- Well-defined resolvent since $\dot{\Omega} \perp \Omega$

Exponential estimates, conclusion

Repeat inductively to obtain

$$\psi_\epsilon^N(s) = \mathrm{e}^{-\frac{\mathrm{i}}{\epsilon}\int_0^s E_s \mathrm{d}r}\left(\varphi^0(s) + \cdots \epsilon^N \varphi^N(s) + \epsilon^{N+1}\varphi_\perp^N(s)\right)$$

- ▷ Key step: Decomposition $\mathrm{span}\{\Omega\} \oplus \mathrm{span}\{\Omega\}^\perp$ with a gap γ
- ▷ Control up to order k requires $H \in C^{k+1}$
- ▷ If $H \in C^\infty$, the error estimate $\mathcal{O}(\epsilon^\infty)$. In fact

$$\left\|\psi_\epsilon - \psi_\epsilon^N\right\| \leq \kappa(\gamma)\mathrm{e}^{-\frac{C(\gamma)}{\epsilon}}$$

G.A. Hagedorn and A. Joye, J. Math. Anal. Appl., 2002

The many-body problem

Recall

$$\|\rho_\epsilon(s) - P_s\| \leq C(\gamma)\,\epsilon \sup_{s\in[0,1]} \|\dot{H}_s\|$$

The energy is extensive: In a volume V,

$$\|H_s\| \sim V$$

and if the driving is uniform

$$\|\dot{H}_s\| \sim V$$

The adiabatic estimate is meaningful only in the few-body regime:

$$\epsilon V \ll 1$$

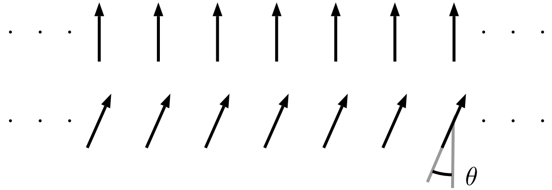
What about the many-body regime $\epsilon \ll 1$ uniformly as $V \to \infty$?

Infrared catastrophe

A non-interacting spin-$1/2$ chain

$$H_{[-L,L],h} = \sum_{x=-L}^{L} h \cdot \sigma_x$$

where σ_x is the vector of Pauli matrices at site x, and $h \in \mathbb{S}^2$



$$\Omega_0 = \cdots \otimes |0\rangle \otimes \cdots \otimes |0\rangle \otimes \cdots$$

$$\Omega_\theta = \cdots \otimes |\theta\rangle \otimes \cdots \otimes |\theta\rangle \otimes \cdots$$

Many-body catastrophe: $\Omega_\theta \perp \Omega_0$ for large L

$$\langle \Omega_\theta, \Omega_0 \rangle = (\cos\theta)^{2L+1} \longrightarrow 0 \qquad (L \to \infty)$$

PART II

Locality in quantum spin systems

Extended systems

Two kinds of many-body quantum systems

- Quantum spin systems (QSS): A countable collection of finite-dimensional quantum systems
- Interacting fermions hopping on a lattice

In both cases:

- The ‘lattice’ is equipped with a metric: notion of locality
- The infinite volume, finite density, limit can be controlled

Here, only QSS. For lattice fermions, see

B. Nachtergaele, R. Sims, A. Young, arXiv:1705.08553

and for adiabatic theorems:

D. Monaco, S. Teufel, arXiv:1707.01852

Quantum spin systems

A countable collection of quantum systems, labelled by $x \in \Gamma$ with finite dimensional Hilbert spaces $\mathcal{H}_x$

Typically:

- ▷ $\Gamma = \mathbb{Z}^d$ equipped with graph distance $d(\cdot, \cdot)$
- ▷ $\mathcal{H}_x = \mathbb{C}^{2S+1}$ the Hilbert space of states of a spin-S

For finite $\Lambda \subset \Gamma$

- ▷ Local Hilbert space

$$\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{H}_x$$

- ▷ The algebra of local observables

$$\mathcal{A}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{L}(\mathcal{H}_x) = \mathcal{L}(\mathcal{H}_\Lambda)$$

Quantum spin systems

Natural identification $\Lambda_1 \subset \Lambda_2 \Rightarrow \mathcal{A}_{\Lambda_1} \subset \mathcal{A}_{\Lambda_2}$ by

$$B \in \mathcal{A}_{\Lambda_1} \longleftrightarrow B \otimes 1_{\Lambda_2 \setminus \Lambda_1} \in \mathcal{A}_{\Lambda_2}$$

An interaction is a map

$$\Phi : Z \longmapsto \Phi(Z) = \Phi(Z)^* \in \mathcal{A}_Z$$

and $\|\Phi(Z)\|$ decays in the diameter of Z

A Hamiltonian is an extensive operator

$$H_\Lambda = \sum_{Z \subset \Lambda} \Phi(Z) \in \mathcal{L}(\mathcal{H}_\Lambda)$$

Note: The decomposition of H_Λ is not unique

Example

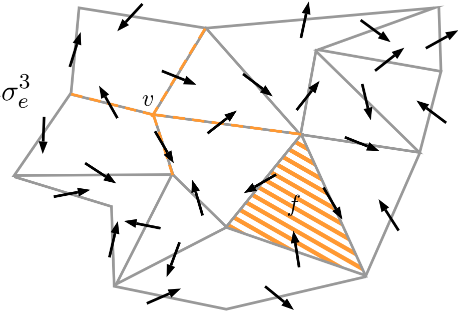
The toric code Hamiltonian

Λ: set of edges

$$H_{\Lambda,h} = -\sum_{v:\text{vertex}} A_v - \sum_{f:\text{face}} B_f - \sum_{e\in\Lambda} h\sigma_e^3$$

where

$$A_v = \prod_{i\in v} \sigma_i^1, \qquad B_f = \prod_{i\in f} \sigma_i^3$$



Can be defined on $\mathbb{Z}^2$
or on cell decompositions of compact 2d surfaces

Topologically ordered ground states, see B. Nachtergaele's talk

Dynamics and states: Heisenberg picture

Finite volume dynamics

$$\tau_{t,\Lambda}(A) = \mathrm{e}^{\mathrm{i}tH_\Lambda} A \mathrm{e}^{-\mathrm{i}tH_\Lambda}$$

The operator H_Λ has no limit as $\Lambda \to \Gamma$, but the dynamics does

$$\tau_t^H(A) = \lim_{\Lambda \to \Gamma} \tau_{t,\Lambda}(A), \quad A \in \mathcal{L}(\mathcal{H}_Z)$$

States are a normalized positive linear functionals

$$A \longmapsto \mathrm{Tr}(\rho_\Lambda A) \in \mathbb{C}$$

with dynamics

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\rho_\Lambda(t) = [H_\Lambda, \rho_\Lambda(t)]$$

Locality I

Let $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$ with $d(X,Y) = d > 0$. Then

$$[\tau_{0,\Lambda}(A), B] = [A, B] = [A \otimes 1_{\Lambda \setminus X}, 1_{\Lambda \setminus Y} \otimes B] = 0\,.$$

Propagation estimate: the Lieb-Robinson bound (LRB)

$$\|[\tau_{t,\Lambda}(A), B]\| \leq C(A,B)\mathrm{e}^{-\mu(d - v|t|)}$$

for constant $\mu, v > 0$.

- Message: For times $|t| \leq d/v$, $\tau_{t,\Lambda}(A)$ almost commutes with B
- Rate μ depends on the decay of Φ
- Key to prove the existence of the limit $\tau_{t,\Lambda}(A)$ as $\Lambda \to \Gamma$

E.H. Lieb and D. Robinson, Commun. Math. Phys., 1972

M.B. Hasting and T. Koma, Commun. Math. Phys., 2006

B. Nachtergaele and R. Sims, Commun. Math. Phys., 2006 & Contemp. Math., 2010

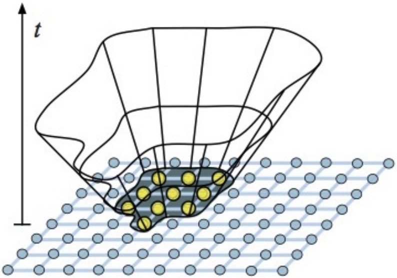
Locality II

The Schrödinger equation has infinite propagation speed, but...

Corollary of the LRB: for any $\delta > 0$ and any $t \in \mathbb{R}$, there exists $A_t^\delta \in \mathcal{A}_{X^{v|t|+\delta}}$ such that

$$\left\|\tau_t(A) - A_t^\delta\right\| \leq C(A)\mathrm{e}^{-\mu\delta}$$

where $X^r := \{x \in \mathbb{Z}^d : \mathrm{dist}(x, X) \leq r\}$



Schuch et. al., Phys. Rev. A, 2011

Almost commuting operators

Let $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. If $A \in \mathcal{L}(\mathcal{H}_1)$ and $B \in \mathcal{L}(\mathcal{H}_2)$, then $[A, B] = 0$.
Also: if $A \in \mathcal{L}(\mathcal{H})$ and

$$[A, B] = 0 \quad \forall B \in \mathcal{L}(\mathcal{H}_2) \qquad \Longrightarrow \qquad A \in \mathcal{L}(\mathcal{H}_1)$$

More generally: If $A \in \mathcal{L}(\mathcal{H})$ is such that

$$\|[A, 1 \otimes B]\| \leq \epsilon \, \|B\|$$

for all $B \in \mathcal{L}(\mathcal{H}_2)$, then there exists $\Pi(A) \in \mathcal{L}(\mathcal{H}_1)$ such that

$$\|\Pi(A) \otimes 1 - A\| \leq 2\epsilon.$$

In other words, almost commutation implies almost localization

Almost commuting operators

Let

$$\Pi(A) = \int_{\mathcal{U}(\mathcal{H}_2)} (1_{\mathcal{H}_1} \otimes U)^* \, A \, (1_{\mathcal{H}_1} \otimes U) \mathrm{d}\mu(U)$$

where μ is the normalized Haar measure $\mu(\mathcal{S}) = \mu(\mathcal{S}U')$, $U' \in \mathcal{U}(\mathcal{H}_2)$

Then

- $\Pi(A) = \Pi(A) \otimes 1_{\mathcal{H}_2}$ because

$$\begin{aligned}\Pi(A)(1_{\mathcal{H}_1} \otimes V) &= \int_{\mathcal{U}(\mathcal{H}_2)} (1_{\mathcal{H}_1} \otimes \tilde{U}V^*)^* \, A \, (1_{\mathcal{H}_1} \otimes \tilde{U}) \mathrm{d}\mu(\tilde{U}V^*) \\ &= (1_{\mathcal{H}_1} \otimes V)\Pi(A)\end{aligned}$$

- Approximation

$$\Pi(A) \otimes 1_{\mathcal{H}_2} - A = \int_{\mathcal{U}(\mathcal{H}_2)} (1_{\mathcal{H}_1} \otimes U)^* [A, (1_{\mathcal{H}_1} \otimes U)] \mathrm{d}\mu(U)$$

Parallel transport

Back to spectral projections P_s and vectors $P_s\Omega_s = \Omega_s$, namely

$$\dot{P}_s\Omega_s = (1 - P_s)\dot{\Omega}_s$$

Motion of Ω_s within $\mathrm{Ran}(P_s)$ free.

A natural choice: No motion, a.k.a. parallel transport

$$P_s\dot{\Omega}_s = 0$$

yields

$$\dot{\Omega}_s = \dot{P}_s\Omega_s = (1 - P_s)\dot{P}_s P_s\Omega_s = [\dot{P}_s, P_s]\Omega_s$$

Kato's flow of ground states

$$\Omega_s = U^{\mathsf{K}}(s)\Omega_0, \qquad \mathrm{i}\dot{U}(s) = \mathrm{i}[\dot{P}_s, P_s]U^{\mathsf{K}}(s)$$

T. Kato, J. Phys. Soc. Japan, 1950

Remarks

Two choices made:

- Choice of flow within $\mathrm{Ran}(P_s)$: $\Omega_0 \mapsto \Omega_s$
- Choice of unitary to implement the flow $\Omega_s = U(s)\Omega_0$

But: the self-adjoint generator

$$G_s^{\mathsf{K}} = \mathrm{i}[\dot{P}_s, P_s]$$

is non-local.

Better (local) choice?

$$P_s = U(s)P_0U(s)^*$$

with generator $G_s = \mathrm{i}\dot{U}(s)U(s)^* = -\mathrm{i}U(s)\dot{U}(s)^*$, namely

$$\dot{P}_s = -\mathrm{i}[G_s, P_s]$$

Quasi-adiabatic flow I

Yes, using the gap! Hastings' generator:

$$G_s^{\mathsf{H}} = \int_{\mathbb{R}} W(t) \mathrm{e}^{\mathrm{i}tH_s} \dot{H}_s \mathrm{e}^{-\mathrm{i}tH_s} \mathrm{d}t$$

where $W \in L^1(\mathbb{R}; \mathbb{R})$

M.B. Hastings, Phys. Rev. Lett, 2004

SB, S. Michalakis, B. Nachtergaele and R. Sims, Commun. Math. Phys. 2012

- Good flow: $[H, P] = 0$ implies $[\dot{H}, P] = [\dot{P}, H]$, hence

$$-\mathrm{i}[G^{\mathsf{H}}, P] = -\mathrm{i} \int_{\mathbb{R}} W(t) \mathrm{e}^{\mathrm{i}tH} [\dot{P}, H] \mathrm{e}^{-\mathrm{i}tH} \mathrm{d}t = \int_{\mathbb{R}} W(t) \frac{\mathrm{d}}{\mathrm{d}t} \tau_t(\dot{P}) \mathrm{d}t$$

  Choice 1: $W(t) = \Theta(t) - 1/2$ yields $\dot{P}$ but divergent

Quasi-adiabatic flow II

$-\mathrm{i}[G^{\mathsf{H}}, P] = \int_{\mathbb{R}} W(t) \frac{\mathrm{d}}{\mathrm{d}t} \tau_t(\dot{P}) \mathrm{d}t$

Choice 2: $W(t) = \Theta(t) - 1/2 + T(t)$ with

$$\widehat{T'}(\xi) = 0 \qquad \text{whenever } |\xi| \geq \gamma.$$

because ($\mathrm{e}^{-\mathrm{i}Ht} = \int \mathrm{e}^{-\mathrm{i}\lambda t} dE(\lambda)$)

$$-\int T'(t) \mathrm{e}^{\mathrm{i}tH} \dot{P} \mathrm{e}^{-\mathrm{i}tH} \mathrm{d}t = -\sqrt{2\pi} \int \widehat{T'}(\mu - \lambda) \mathrm{d}E(\lambda) \dot{P} \mathrm{d}E(\mu) = 0$$

since $\dot{P}$ is off-diagonal

- Parallel transport: If $\widehat{W}(0) = 0$ ie W is odd, then

$$PG^{\mathsf{H}}P = \int W(t) P\dot{H}P \mathrm{d}t = \sqrt{2\pi}\widehat{W}(0) P\dot{H}P = 0$$

Quasi-adiabatic flow: Locality

Finally assume fast decay

$$W(t) = \mathcal{O}\left(|t|^{-\infty}\right)$$

Then for $A \in \mathcal{A}_X$

$$\int W(t)\tau_t(A)\mathrm{d}t$$

is almost local:

- ▷ For $|t| \leq T$, $\tau_t(A)$ is supported near X
- ▷ For $|t| \geq T$, use the decay of W

Hence

$$G^{\mathsf{H}} = \int W(t)\tau_t(\dot{H})\mathrm{d}t = \sum_Z \int W(t)\tau_t(\dot{\Phi}(Z))\mathrm{d}t$$

is a sum of almost local terms, a.k.a. a Hamiltonian

Summary and perspective

- Both $G^{\mathsf{K}}, G^{\mathsf{H}}$ generate parallel transport
- G^{K} has no good locality properties
- G^{H} generates U^{H} satisfying a Lieb-Robinson bound: suited for many-body applications
- So far: no adiabatic dynamics (no ϵ), but anticipating:

  U_s^{H} is the zeroth order of the local adiabatic expansion of the Schrödinger unitary $U_\epsilon(s)$

PART III

Many-body adiabatic theorem

Recap: the goal

Time dependent quantum spin Hamiltonian in volume Λ

$$H_{\Lambda,s} = \sum_{Z\subset\Lambda} \Phi_s(Z)$$

with gapped spectral projection $P_{\Lambda,s}$

Dynamics

$$\begin{cases} \mathrm{i}\epsilon\dot{\psi}_{\Lambda,\epsilon}(s) = H_{\Lambda,s}\psi_{\Lambda,\epsilon}(s), & s \in [0,1] \\ \psi_{\Lambda,\epsilon}(0) = \Omega_{\Lambda,0} \end{cases}$$

where $\Omega_{\Lambda,0} = P_{\Lambda,0}\Omega_{\Lambda,0}$

Goal: control the long time dynamics

$$\psi_{\Lambda,\epsilon}(s) - \Omega_{\Lambda,s} = \mathcal{O}(\epsilon), \quad s \in [0,1]$$

uniformly in the volume. But in which sense?

Local changes

Uniform topology is too strong (infrared catastrophe)

Test $\psi_{\Lambda,\epsilon}(s)$ only locally:

$$\langle \psi_{\Lambda,\epsilon}(s), O\psi_{\Lambda,\epsilon}(s)\rangle - \langle \tilde{\psi}_{\Lambda,\epsilon}(s), O\tilde{\psi}_{\Lambda,\epsilon}(s)\rangle = \mathcal{O}(\epsilon)$$

where

$$\tilde{\psi}_{\epsilon}(s) = P_s \tilde{\psi}_{\epsilon}(s)$$

and O is local, arbitrary but fixed

No uniformity in O, expect error bounds to depend on

$$\|O\|, \quad |\mathrm{supp}(O)|,$$

but not on Λ

Many-body adiabatic theorem

Spectral gap: With $E_{\Lambda,s} := \inf \mathrm{Spec}(H_{\Lambda,s})$,

$$\gamma := \inf_{\Lambda\in\mathcal{F}(\Gamma), s\in[0,1]} \sup\{\delta : (E_{\Lambda,s}, E_{\Lambda,s}+\delta)\cap \mathrm{Spec}(H_{\Lambda,s}) = \emptyset\}$$

Theorem [SB-Fraas-De Roeck]
Let Γ be d-dimensional. If the Hamiltonian is

- *gapped: $\gamma > 0$,*
- *smooth: $H_s \in \mathcal{C}^{d+1}([0,1])$*

then there is $\tilde\psi_{\Lambda,\epsilon}(s) = P_{\Lambda,s}\tilde\psi_{\Lambda,\epsilon}(s)$ such that

$$\left|\left\langle \psi_{\Lambda,\epsilon}(s), O\psi_{\Lambda,\epsilon}(s)\right\rangle - \left\langle \tilde\psi_{\Lambda,\epsilon}(s), O\tilde\psi_{\Lambda,\epsilon}(s)\right\rangle\right| \leq C|\mathrm{supp}(O)|^2\|O\|\epsilon$$

for all $\Lambda\in\mathcal{F}(\Gamma)$, with C independent of Λ

On parallel transport

If $\mathrm{Ran}(P_{\Lambda,s})$ is nearly degenerate of width

$$w \leq C \min\left\{\epsilon^2, \frac{\epsilon}{|\Lambda|}\right\}$$

Then

$$\tilde{\psi}_{\Lambda,\epsilon}(s) = \Omega_\Lambda(s)$$

is parallel transported

$$\mathrm{i}\dot{\Omega}_\Lambda(s) = \mathrm{i}[\dot{P}_{\Lambda,s}, P_{\Lambda,s}]\Omega_\Lambda(s)$$

and independent of ϵ

Locality again

Looking at expectation values of local observables has

- ▷ An advantage: Essentially ignores Λ
- ▷ A disadvantage: Recall

$$\rho_\epsilon(s) - P_s = \epsilon \int_0^s \sigma^{s,s'} \left(\frac{\mathrm{d}}{\mathrm{d}s'} (L_{s'}^{-1} \dot{P}_{s'}) \right) \mathrm{d}s'$$

$\sigma^{s,s'}$ is norm preserving
but it spreads local observables over volume

$$|s' - s|^d = \epsilon^{-d} |t' - t|^d$$

yielding a naive bound

$$\mathcal{O}(\epsilon^{1-d}) \quad \text{as} \quad \epsilon \to 0$$

Solution: Expand to higher order in adiabatic perturbation theory

The map $\mathcal{I}$

Define $\mathcal{I}_s : \mathcal{A} \to \mathcal{A}$ by

$$\mathcal{I}_s(A) := \int_{\mathbb{R}} W(t) \mathrm{e}^{\mathrm{i}tH_s} A \mathrm{e}^{-\mathrm{i}tH_s} \mathrm{d}t$$

where $W \in L^1(\mathbb{R})$ decays faster than any inverse power and such that

$$\widehat{W}(\xi) = \frac{-\mathrm{i}}{\sqrt{2\pi}\xi}, \quad \text{if} \quad |\xi| \geq \gamma.$$

Fact: If A is local , then $\mathcal{I}_s(A)$ is almost local

- for short times: $\tau_t(A)$ almost local by LRB
- for long times: $W(t)\|\tau_t(A)\| = W(t)\|A\|$ vanishes by decay of W

Local spectral flow

Claim: $\mathcal{I}(\cdot)$ is a local inverse of $-\mathrm{i}[H,\cdot]$

$$A = -\mathrm{i}[H, \mathcal{I}(A)]$$

whenever

$$A = PA(1-P) + (1-P)AP$$

Indeed

$$\mathcal{I}(A) = \int W(t) \mathrm{e}^{\mathrm{i}\lambda t} \mathrm{e}^{-\mathrm{i}\mu t} \mathrm{d}E_\lambda A \mathrm{d}E_\mu \mathrm{d}t = \sqrt{2\pi} \int \widehat{W}(\mu - \lambda) \mathrm{d}E_\lambda A \mathrm{d}E_\mu$$

and

$$\begin{aligned} -\mathrm{i}[H, \mathcal{I}(A)] &= -\mathrm{i}\sqrt{2\pi} \int_{|\lambda-\mu|\geq\gamma} (\lambda - \mu) \widehat{W}(\mu - \lambda) \mathrm{d}E_\lambda A \mathrm{d}E_\mu \\ &= -\mathrm{i}\sqrt{2\pi} \int_{|\lambda-\mu|\geq\gamma} (\lambda - \mu) \frac{\mathrm{i}}{\sqrt{2\pi}(\lambda - \mu)} \mathrm{d}E_\lambda A \mathrm{d}E_\mu = A \end{aligned}$$

Quasi-adiabatic evolution, revisited

For a spectral projection

$$\dot{P}_s = P_s \dot{P}_s (1 - P_s) + (1 - P_s) \dot{P}_s P_s$$

Hence

$$\dot{P}_s = -\mathrm{i}[H_s, \mathcal{I}_s(\dot{P}_s)] = -\mathrm{i}\mathcal{I}_s([H_s, \dot{P}_s]) = \mathrm{i}\mathcal{I}_s([\dot{H}_s, P_s])$$

namely

$$\dot{P}_s = \mathrm{i}[\mathcal{I}_s(\dot{H}_s), P_s]$$

and

$$G_s^{\mathsf{H}} = \mathcal{I}_s(\dot{H}_s) = \sum_Z \mathcal{I}_s(\dot{\Phi}_s(Z))$$

is a sum of almost local terms

Local spectral flow

Another consequence: For any B, $[B, P]$ is off-diagonal, hence

$$\begin{aligned}[B,P] &= -\mathrm{i}\big[H, \mathcal{I}([B,P])\big] \\ &= -\mathrm{i}\big[H, [\mathcal{I}(B), P]\big] \\ &= -\mathrm{i}\big[[H, \mathcal{I}(B)], P\big]\end{aligned}$$

by Jacobi's identity and

$$[H, P] = 0$$

The dressed states

Key idea:
Compare the evolved $\rho_\epsilon(s)$ with a dressed ground state projection

$$\Pi_\epsilon(s) = V_\epsilon(s) P_s V_\epsilon(s)^*, \qquad V_\epsilon(s) = \mathrm{e}^{\mathrm{i}\epsilon A_1(s) + \epsilon^2 A_2(s) + \cdots}$$

such that A_j's are extensive operators, $\Pi_\epsilon(s)$ solves

$$\mathrm{i}\epsilon \dot{\Pi}_\epsilon(s) = [H_s + R_\epsilon(s), \Pi_\epsilon(s)]$$

where $R_\epsilon(s)$ is $\mathcal{O}(\epsilon^n)$
Then, when tested against a local observable:

- $\rho_\epsilon(s) - \Pi_\epsilon(s)$ is of order ϵ^{n-d} since

$$\mathrm{i}\epsilon \dot{\rho}_\epsilon = [H_s, \rho_\epsilon(s)]$$

- $\Pi_\epsilon(s) - P_s$ is of order ϵ

Adiabatic expansion I

Ansatz $\Pi = VPV^*$

$$\mathrm{i}\epsilon\dot{\Pi} = [H, \Pi] + V\left[\mathrm{i}\epsilon V^*\dot{V} - \epsilon\mathcal{I}(\dot{H}) + (H - V^*HV), P\right]V^*$$

with

$$V = \mathrm{e}^{\mathrm{i}\epsilon A_1 + \epsilon^2 A_2 + \cdots}$$

Now

$$H - V^*HV = \mathrm{i}\epsilon[A_1, H] + \mathcal{O}(\epsilon^2)$$

and

$$\dot{V} = \mathcal{O}(\epsilon)$$

No zeroth order: The zeroth order dynamics is fully within $\mathrm{Ran}P$

Adiabatic expansion II

First order is zero if

$$\mathrm{i}\big[[A_1, H], P\big] = \big[\mathcal{I}(\dot{H}), P\big]$$

to be solved for A_1. By the local inverse lemma

$$A_1 = \mathcal{I}\left(\mathcal{I}(\dot{H})\right)$$

Continue inductively: By choosing $A_1, \ldots A_n$, the remainder

$$V^*RV = \mathrm{i}\epsilon V^*\dot{V} - \epsilon\mathcal{I}(\dot{H}) + H - V^*HV$$

is of order ϵ^{n+1}.

In which sense? All A_j's are local extensive operators, so that

$$|\mathrm{Tr}(V^*RVO)| \leq C\|O\||\mathrm{supp}(O)|\epsilon^{n+1}$$

Proof I

Starting from a $\Omega_0 \in \mathrm{Ran} P_0$, define

$$\phi_{n,\epsilon}(s) = V_{n,\epsilon}(s,0)\Omega_0$$

where

$$\mathrm{i}\epsilon \dot{V}_{n,\epsilon}(s,s') = (H_s + R_{n,\epsilon})V_{n,\epsilon}(s,s'), \qquad V_{n,\epsilon}(s',s') = 1$$

Then

$$\begin{aligned}\langle \psi_\epsilon(s), O\psi_\epsilon(s)\rangle &- \langle \phi_{n,\epsilon}(s), O\phi_{n,\epsilon}(s)\rangle \\ &= -\frac{\mathrm{i}}{\epsilon}\int_0^s \langle \phi_{n,\epsilon}(r), \left[R_{n,\epsilon}(r), U_\epsilon(s,r)^* O U_\epsilon(s,r)\right]\phi_{n,\epsilon}(r)\rangle \mathrm{d}r \\ &= \mathcal{O}(\epsilon^{-1+(n+1)-d})\end{aligned}$$

Proof II

By construction

$$\phi_{n,\epsilon}(s) \in \mathrm{Ran}(\Pi_{n,\epsilon}(s))$$

so that there is $\tilde{\psi}_{n,\epsilon}(s) \in \mathrm{Ran} P_s$ such that

$$\langle \tilde{\psi}_{n,\epsilon}(s), O \tilde{\psi}_{n,\epsilon}(s) \rangle - \langle \phi_{n,\epsilon}(s), O \phi_{n,\epsilon}(s) \rangle = \mathcal{O}(\epsilon)$$

Summarizing:

- $\psi_\epsilon(s)$ the dynamically evolved vector
- $\phi_{n,\epsilon}(s)$ a ‘dressed’ state
- $\tilde{\psi}_{n,\epsilon}(s)$ a state in $\mathrm{Ran} P_s$

with

$$\psi_\epsilon(s) = \phi_{n,\epsilon}(s) + \mathcal{O}(\epsilon^{n-d}), \qquad \phi_{n,\epsilon}(s) = \tilde{\psi}_{n,\epsilon}(s) + \mathcal{O}(\epsilon)$$

Choose $n \geq d+1$

Remarks

- Differentiability assumption

$$A_1(s) = \mathcal{I}_s(\mathcal{I}_s(\dot{H}_s))$$

Generally: $A_n(s)$ depends on $H_s^{(k)}$, for $1 \leq k \leq n$

- Local-in-time dependence: If $H_{s=1}^{(k)} = 0$ for $1 \leq k \leq n$, then

$$\Pi_{n,\epsilon}(1) - P_1 = \mathcal{O}(\epsilon^n)$$

yielding an improved adiabatic estimate: e.g. $\mathcal{O}(\epsilon^\infty)$ if $H \in C^\infty$

- Parallel transport:The dynamics of $\tilde{\psi}_{n,\epsilon}(s)$ is generated by

$$\mathrm{i}[A_1, H]P = \mathrm{i}[[A_1, H], P]P = \mathrm{i}[\mathcal{I}(\dot{H}), P]P = \mathrm{i}[G^{\mathsf{H}}, P]P$$

to $\mathcal{O}(\epsilon)$

On the gap assumption

Crucial assumption: open spectral gap, uniformly in Λ, s

- When can it be proved?
  Perturbations of frustration-free systems
  S. Bravyi, M.B. Hasting and S. Michalakis, J. Math. Phys., 2010
  S. Michalakis and J. Zwolak, Commun. Math. Phys., 2013

  Perturbations of non-interacting fermion systems
  A. Giuliani, V. Mastropietro, M. Porta, Commun. Math. Phys., 2016
  M.B. Hasting, arXiv:1706.02270
  W. De Roeck and M. Salmhofer, arXiv:1712.00977
  B. Nachtergaele, R. Sims, A. Young, arXiv:1705.08553
- What about gapless (disordered) systems?

PART IV

Linear response theory

Motivation

Ohm's law of electric conductivity, Fourier's law of thermal conductivity

General form

$$J = \chi F$$

J is a current
F is a force
χ is the matrix of linear response coefficients

Three connected mathematical problems:

- ▷ Validity of the linear relation
- ▷ Derivation of a general formula for χ
- ▷ Computation of χ in a given setting

One famous example (see A. Bols' talk):
The quantisation of the quantum Hall conductivity

State-of-the-art

In Physics: Well understood

- ▷ Derivation by the first order perturbation theory
- ▷ Kubo formula (1957)
- ▷ Some doubts about the order of limits, van Kampen (1971)

In Math: Very few results

- ▷ One of the open problems on Simon’s 1984 list
- ▷ Non-interacting gapped systems (Elgart, Schlein 2004)
- ▷ Disorded systems (Bouclet et al 2005)
- ▷ Smoothed in frequency (Klein et al 2007, Bru, Pedra 2017)
- ▷ Compactly supported driving (Jakšić et al 2006)

Linear response: Setting

Setting: An adiabatically switched on perturbation

$$H = H_{\mathrm{i}} + \mathrm{e}^{\epsilon t}\alpha V, \qquad t \in (-\infty, 0]$$

with $\alpha \in \mathbb{R}$. Also,

$$P_{\mathrm{i}} = P\{\alpha = 0, s\} = P_{\{\alpha, t=-\infty\}}$$

the ground state projection of H_{i}

Linear response coefficient of a local observable J:

$$\chi_{J,V} := \lim_{\alpha\to 0}\lim_{\epsilon\to 0}\lim_{\Lambda\to\Gamma}\frac{\mathrm{Tr}(J\rho_{\alpha,\epsilon}(0)) - \mathrm{Tr}(JP_{\mathrm{i}})}{\alpha}$$

Example: V is an electromotive force
J is an electric current
$\chi_{J,V}$ is the conductivity

Formal derivation I

Initial condition

$$\lim_{t\to-\infty} \mathrm{e}^{\mathrm{i}H_\mathrm{i}t}\rho_{\epsilon,\alpha}(t)\mathrm{e}^{-\mathrm{i}H_\mathrm{i}t} = P_\mathrm{i}$$

For $\varrho(t) := \mathrm{e}^{\mathrm{i}H_\mathrm{i}t}\rho_{\epsilon,\alpha}(t)\mathrm{e}^{-\mathrm{i}H_\mathrm{i}t}$,

$$\varrho_{\epsilon,\alpha}(t) = \varrho_{\epsilon,\alpha}(t_0) - \mathrm{i}\alpha\int_{t_0}^{t} \mathrm{e}^{\epsilon\tau}\left[\mathrm{e}^{\mathrm{i}H_\mathrm{i}\tau}V\mathrm{e}^{-\mathrm{i}H_\mathrm{i}\tau}, \varrho_{\epsilon,\alpha}(\tau)\right]\mathrm{d}\tau$$

Let $t = 0, t_0 \to -\infty$,

$$\rho_{\epsilon,\alpha}(0) - P_\mathrm{i} = -\mathrm{i}\alpha\int_{-\infty}^{0} \mathrm{e}^{\epsilon\tau}\mathrm{e}^{\mathrm{i}H_\mathrm{i}\tau}\left[V, \rho_{\epsilon,\alpha}(\tau)\right]\mathrm{e}^{-\mathrm{i}H_\mathrm{i}\tau}\mathrm{d}\tau$$

to obtain to first order in α (only linear response)

$$\mathrm{Tr}\big(J(\rho_{\alpha,\epsilon}(0) - P_\mathrm{i})\big) = -\mathrm{i}\alpha\int_{-\infty}^{0} \mathrm{e}^{\epsilon\tau}\mathrm{Tr}\big(P_\mathrm{i}\left[\mathrm{e}^{-\mathrm{i}H_\mathrm{i}\tau}J\mathrm{e}^{\mathrm{i}H_\mathrm{i}\tau}, V\right]\big)\mathrm{d}\tau$$

Formal derivation II

We obtained Kubo's formula

$$\chi^{\mathsf{Kubo}}_{J,V} = \lim_{\epsilon\to 0} \mathrm{i} \int_{-\infty}^{0} \mathrm{e}^{\epsilon\tau} \mathrm{Tr}\big(P_{\mathrm{i}} \left[V, \mathrm{e}^{-\mathrm{i}H_{\mathrm{i}}\tau} J \mathrm{e}^{\mathrm{i}H_{\mathrm{i}}\tau}\right]\big) \mathrm{d}\tau$$

The catch:

$$\big\| [V, \mathrm{e}^{-\mathrm{i}H_{\mathrm{i}}\tau} J \mathrm{e}^{\mathrm{i}H_{\mathrm{i}}\tau}] \big\| \sim \tau^d$$

and since

$$\int e^{\epsilon\tau} \tau^d \mathrm{d}\tau \sim \frac{1}{\epsilon^{d+1}}$$

the limit $\epsilon \to 0$ may not exist

Adiabaticity and the spectral flow

- By the adiabatic theorem

$$\rho_{\alpha,\epsilon}(0) - P_{\{\alpha,s=0\}} \longrightarrow 0$$

  locally as $\epsilon \to 0$, uniformly in Λ
- Since $H_\alpha = H_{\mathrm{i}} + \alpha V$,

$$\partial_\alpha H_\alpha = V$$

  so that the spectral flow yields

$$\lim_{\alpha\to 0} \alpha^{-1}(P_{\{\alpha,s=0\}} - P_{\mathrm{i}}) = \mathrm{i}[\mathcal{I}_{\mathrm{i}}(V), P_{\mathrm{i}}]$$

  uniformly in Λ

The linear response coefficient

Hence ($\overline{\mathrm{Tr}} = \sup_\Lambda \mathrm{Tr}$)

$$\begin{aligned}
&\alpha^{-1}\overline{\mathrm{Tr}}\big(J(\rho_{\alpha,\epsilon}(0) - P_{\mathrm{i}})\big) \\
&\quad \leq \alpha^{-1}\overline{\mathrm{Tr}}\big(J(\rho_{\alpha,\epsilon}(0) - P_{\{\alpha,s=0\}})\big) + \alpha^{-1}\overline{\mathrm{Tr}}\big(J(P_{\{\alpha,s=0\}} - P_{\mathrm{i}})\big) \\
&\qquad \longrightarrow \alpha^{-1}\overline{\mathrm{Tr}}\big(J(P_{\{\alpha,s=0\}} - P_{\mathrm{i}})\big) \qquad (\epsilon \to 0) \\
&\qquad \longrightarrow \mathrm{i}\overline{\mathrm{Tr}}\big(J[\mathcal{I}_{\mathrm{i}}(V), P_{\mathrm{i}}]\big) \qquad (\alpha \to 0)
\end{aligned}$$

Theorem. [SB-Fraas-de Roeck]
If $H = H_{\mathrm{i}} + \beta V$ is gapped for in a neighbourhood of $\beta = 0$, then

$$\chi_{J,V} = -\mathrm{i}\overline{\mathrm{Tr}}\big(P_{\mathrm{i}}[\mathcal{I}_{\mathrm{i}}(V), J]\big)$$

And again: $\chi_{J,V}$ expressed only in terms of ‘unperturbed’ quantities

Kubo?

Kubo's formula, again:

$$\chi_{J,V}^{\text{Kubo}} = \lim_{\epsilon\to 0} \mathrm{i} \int_{-\infty}^{0} \mathrm{e}^{\epsilon\tau} \mathrm{Tr}\big(P\left[V, \mathrm{e}^{-\mathrm{i}H\tau} J \mathrm{e}^{\mathrm{i}H\tau}\right]\big)\mathrm{d}\tau$$

Replace $V \to \tilde{V} = PV(1-P) + (1-P)VP$ and use $\tilde{V} = -\mathrm{i}[H, \mathcal{I}(\tilde{V})]$
Note

$$\mathrm{i}\int_0^\infty e^{-\epsilon\tau}\tau_\tau([H,\mathcal{I}(\tilde{V})])\mathrm{d}\tau = \int \frac{(\mu-\lambda)}{(\mu-\lambda)-\mathrm{i}\epsilon}\mathrm{d}E(\lambda)\mathcal{I}(\tilde{V})\mathrm{d}E(\mu) \longrightarrow \mathcal{I}(\tilde{V})$$

as $\epsilon \to 0$, because there is a gap and $\mathcal{I}(\tilde{V})$ is off-diagonal. Hence

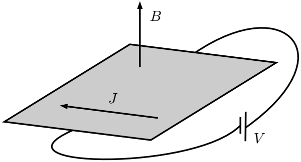
$$\chi_{J,V}^{\text{Kubo}} = -\mathrm{i}\mathrm{Tr}\big(P\left[\mathcal{I}(V), J\right]\big) = \chi_{J,V}$$

Adiabatic curvature and QHE I

Quantum Hall system: Two-parameter family

$$H(\phi_1, \phi_2), \qquad \phi_1, \phi_2 \in [0, 2\pi)$$

corresponding to an EMF in the 1 direction driving a current in the 2 direction



The electric potential:

$$V = Q_1$$

minimally coupled:

$$-\mathrm{i}[H, V] = \partial_{\phi_1} H$$

The local current:

$$J = \partial_{\phi_2} H$$

Adiabatic curvature and QHE II

Since W is an odd function

$$\chi_{J,V} = -\mathrm{i}\mathrm{Tr}\big(P[\mathcal{I}(\tilde{V}), J]\big) = \mathrm{i}\mathrm{Tr}\big(P[\tilde{V}, \mathcal{I}(J)]\big)$$

Using again $\tilde{V} = -\mathrm{i}[H, \mathcal{I}(\tilde{V})] = -\mathrm{i}\mathcal{I}([H, \tilde{V}])$,

$$\chi_{J,V} = \mathrm{i}\mathrm{Tr}\big(P\,[\mathcal{I}(\partial_{\phi_1} H), \mathcal{I}(\partial_{\phi_2} H)] = \mathrm{i}\mathrm{Tr}\big(P[G_1^{\mathsf{H}}, G_2^{\mathsf{H}}]\big)$$

and since $P\mathcal{I}(\partial_{\phi_j} H)P = 0$,

$$\chi_{J,V} = \mathrm{i}\mathrm{Tr}\big(P\,[[\mathcal{I}(\partial_{\phi_1} H), P], [\mathcal{I}(\partial_{\phi_2} H), P]]\,\big) = \mathrm{i}\mathrm{Tr}\big(P[\partial_{\phi_1} P, \partial_{\phi_2} P]\big)$$

which is the adiabatic curvature, see

J. Avron, R. Seiler and B. Simon, Phys. Rev. Lett., 1983

Summary

Assumption: Spectral gap

- ▷ Infrared catastrophe: Norm estimates fail for extended systems
- ▷ Change topology: consider only local expectation values
- ▷ Both Schrödinger dynamics and Hastings' parallel transport are local
- ▷ Propagation still yields error $\mathcal{O}(\epsilon^{-d})$
- ▷ Solution: Go to higher order in adiabatic perturbation theory to get a many-body adiabatic theorem
- ▷ Crucially, all estimates are independent of the volume: derive validity of linear response theory

Another application of Hastings' local flow:

Quantisation of conductance in many-body systems, see

M.B. Hastings and S. Michalakis, Commun. Math. Phys., 2015

SB, A. Bols, W. De Roeck and M. Fraas, Annales H. Poincaré, 2018