Diffusion for a Markov, Divergence-form Generator

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Abstract

We consider the long-time evolution of solutions to a Schrödinger-type wave equation on a lattice with a Markov random generator. We show that solutions to this problem possess a diffusive scaling limit and compute higher moments. Based on joint work with Jeffrey Schenker.
Statement of the Theorem

**Theorem**

If \( \psi_t \in \ell^2(\mathbb{Z}^d) \) satisfies

\[
\begin{align*}
  i \partial_t \psi_t(x) & = \nabla^\dagger \omega(t) \nabla \psi_t(x) \\
  \psi_0(x) & = \delta_0(x)
\end{align*}
\]

then

\[
\lim_{\eta \to 0^+} \sum_{x \in \mathbb{Z}^d} e^{i \sqrt{\eta} k \cdot x} \mathbb{E} \left( |\psi_t/\eta(x)|^2 \right) = e^{-4t \sum_{e_1, e_2} (k \cdot e_1)(k \cdot e_2) D_{e_1, e_2}}.
\]
What do we mean by diffusion?

- Consider the standard heat equation

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\begin{align*}
\partial_t u(x, t) &= \Delta u(x, t) & (x, t) \in \mathbb{R}^d \times \mathbb{R}^+ \\
u(x, 0) &= \delta_0(x) & x \in \mathbb{R}^d
\end{align*}
\]

with solution \( u(x, t) = (2\pi t)^{-d/2} e^{-|x|^2/4t} \).

- \( x \mapsto c_t u(x, t) \) is a p.d.f. on \( \mathbb{R}^d \) with \( c_t = \left( \int_{\mathbb{R}^d} u(x, t) \, dt \right)^{-1} \) the normalizing constant.
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The $p^{th}$ moment of position is given by

$$\int_{\mathbb{R}^d} |x|^p c_t u(x, t) \, dx = \frac{c_t \omega_d}{(2\pi t)^{d/2}} \int_0^\infty r^{p+d-1} e^{-\frac{r^2}{4t}} \, dr,$$

where $\omega_d = |\partial B(0, 1)|$ is the surface area of the unit ball in $\mathbb{R}^d$.

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**Definition: Diffusive Scaling**

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\begin{align*}
  t & \mapsto \frac{1}{\eta} t \\
  x & \mapsto \frac{1}{\sqrt{\eta}} x
\end{align*}
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as \( \eta \to 0^+ \)

Question: The problem under consideration is defined on the lattice \( \mathbb{Z}^d \). How do we scale a discrete space?

Answer: Mollify.
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Characterization of Diffusion for a Discrete Problem

- \( h \in C_c^\infty(\mathbb{R}^d), \int h \, dx = 1, \ h \geq 0 \)

- Under diffusive scaling, if the convolution \( h \ast |\psi_t|^2 \) converges (weakly) to a solution of the heat equation, then we say that the model exhibits *diffusion*.

- A Fourier transform removes the mollifier from our diffusion criterion.

- **Diffusion Criterion:**

\[
\sum_{x \in \mathbb{Z}^d} e^{i \sqrt{\eta} k \cdot x} |\psi_t/\eta(x)|^2 \to e^{-Dt|k|^2}, \quad k \in \mathbb{T}^d
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Resolvent Analysis

Key step:
\[ \sum_{x \in \mathbb{Z}^d} e^{i \sqrt{\eta} k \cdot x} \mathbb{E}(|\psi_t/\eta(x)|^2) \]

\[= -\frac{1}{2\pi i} \int_{\Gamma} e^{-t z} \left\langle \delta_0 \otimes 1, \frac{\eta}{i \hat{L} \sqrt{\eta} k + B - \eta z} \delta_0 \otimes 1 \right\rangle \, dz \]

Notes:
- The LHS is (almost) the diffusion criterion.
- The expectation allows us to use a Feynman-Kac-Pillet formula.
- FKP allows us to express the expectation as a matrix element of the semigroup \( e^{-t (i \hat{L} \sqrt{\eta} k + B)} \),
- which can be understood by the holomorphic functional calculus:

\[ e^{t (i \hat{L} \sqrt{\eta} k + B)} = \frac{1}{2\pi i} \int_{\Gamma} e^{t z} \frac{1}{i \hat{L} \sqrt{\eta} k + B - z} \, dz \]
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- We have reduced the problem to understanding:

\[ \lim_{\eta \to 0^+} \left\langle \delta_0 \otimes 1, \frac{\eta}{i\hat{L}\sqrt{\eta}k + B - \eta z} \delta_0 \otimes 1 \right\rangle. \]

- From here,
  - use projections and the Schur complement formula.
  - construct a symmetric operator \( D_k \), which is a lower bound for the matrix element in question. Use this to show the limit exists and is of the desired form.

- Higher Moments?

\[ \lim_{\eta \to 0^+} \sum_{x \in \mathbb{Z}^d} e^{i\sqrt{\eta}k \cdot x} \mathbb{E} \left( |\psi_t/\eta(x)|^2 \right) = e^{-4t \sum_{e_1, e_2} (k \cdot e_1)(k \cdot e_2) D_{e_1, e_2}}, \]

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Thank you!

- **Klaus-Jochen Engel and Rainer Nagel.**
  *One-parameter semigroups for linear evolution equations, volume 194 of Graduate Texts in Mathematics.*

- **Yang Kang and Jeffrey Schenker.**
  Diffusion of wave packets in a markov random potential.
  *Journal of Statistical Physics, 134*:1005–1022, 1005.

- **Claude-Alain Pillet.**
  Some results on the quantum dynamics of a particle in a markovian potential.
Thank you!

