Spectral Theory of a Canonical System

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A canonical system is a family of differential equations of the form

\[ Ju'(x) = zH(x)u(x), \quad z \in \mathbb{C}. \]  \hspace{1cm} (1)

\[ J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \] and \( H(x) \) is a 2 x 2 positive semidefinite matrix. Assume \( H \) does not vanish on any open interval.

Consider the Hilbert space

\[ L^2(H, \mathbb{R}_+) = \left\{ f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} : \|f\| < \infty \right\} \]

with inner product \[ \langle f, g \rangle = \int_0^\infty f(x)^* H(x)g(x)dx \]
A linear relation $\mathcal{R} = \{(f, g) : f, g \in \mathcal{H}\}$ on $\mathcal{H}$ is a subspace of $\mathcal{H}^2$.

Domain: $D(\mathcal{R}) = \{f \in \mathcal{H} : (f, g) \in \mathcal{R}\}$

Range: $R(\mathcal{R}) = \{g \in \mathcal{H} : (f, g) \in \mathcal{R}\}$

Inverse: $\mathcal{R}^{-1} = \{(g, f) : (f, g) \in \mathcal{R}\}$

Adjoint of $\mathcal{R}$ in $\mathcal{H}^2$:

$\mathcal{R}^* = \{(h, k) \in \mathcal{H}^2 : \langle f, k \rangle = \langle g, h \rangle \text{ for all } (f, g) \in \mathcal{R}\}$. 
A linear relation $S$ is called symmetric if $S \subset S^*$ and self-adjoint if $S = S^*$.

Goal: Discuss the spectrum of such self adjoint relation. Let $(z - \mathcal{R}) = \{(f, zf - g) : (f, g) \in \mathcal{R}\}$, $\mathcal{R}_z = \mathcal{R}(z - \mathcal{R})$ and $N(\mathcal{R}, z) = \{f : (f, zf) \in \mathcal{R}\}$. Observe that,

$$N(\mathcal{R}^*, \bar{z}) = \mathcal{R}_z^\perp.$$

The regularity domain of $\mathcal{R}$ is the set

$$\Gamma(\mathcal{R}) = \left\{ z \in \mathbb{C} : \exists C(z) > 0 : \|zf - g\| \geq C(z)\|f\|, \forall (f, g) \in \mathcal{R} \right\}.$$
\( \Gamma(\mathcal{R}) \) satisfies the following properties:

1. \( z \in \Gamma(\mathcal{R}) \) if and only if \( (z - \mathcal{R})^{-1} \) is a bounded linear operator on \( D(\mathcal{R}) \).

2. If \( \mathcal{R} \) is symmetric, then \( \mathbb{C} - \mathbb{R} \subset \Gamma(\mathcal{R}) \).

3. \( \Gamma(\mathcal{R}) \) is open.

\( \beta(\mathcal{R}, z) = \dim \mathcal{R}_z^\perp \) is called the defect index of \( \mathcal{R} \) and \( z \).

**Theorem 1**

*The defect index \( \beta(\mathcal{R}, z) \) is constant on each connected subset of \( \Gamma(\mathcal{R}) \). If \( \mathcal{R} \) is symmetric, then the defect index is constant in the upper and lower half-planes.*
For $z \in \mathbb{C}^+$, $m = \beta(\mathcal{R}, z)$ and for $w \in \mathbb{C}^-$, $n = \beta(\mathcal{R}, w)$ are written as a pair $(m, n)$, called the defect indices of $\mathcal{R}$.

**Theorem 2**

Let $\mathcal{R}$ be a closed symmetric relation on a Hilbert space $\mathcal{H}$ with defect indices $(m, n)$ then

1. $\mathcal{R}$ possess self-adjoint extension if and only if its defect indices are equal ($m = n$).

2. A symmetric extension $\mathcal{R}'$ of $\mathcal{R}$ is self-adjoint if and only if $\mathcal{R}'$ is an $m$–dimensional extension of $\mathcal{R}$.
The resolvent set for a closed relation $\mathcal{R}$ is a set

$$\rho(\mathcal{R}) = \left\{ z \in \mathbb{C} : \exists T \in B(\mathcal{H}) : \mathcal{R} = \{(Tf, zTf - f) : f \in \mathcal{H}\} \right\}$$

and the spectrum of $\mathcal{R}$ is

$$\sigma(\mathcal{R}) = \mathbb{C} - \rho(\mathcal{R})$$

We call $S(\mathcal{R}) = \mathbb{C} - \Gamma(\mathcal{R})$ the spectral kernel of $\mathcal{R}$.
Theorem 3

Let $\mathcal{T}$ is a self-adjoint relation on $\mathcal{H}$. Suppose $z \in \Gamma(\mathcal{T})$ and $T = (\mathcal{T} - z)^{-1}$ then

1. $S(\mathcal{T}) = \sigma(\mathcal{T})$
2. If $\lambda \in \Gamma(T)$ then $(z - \frac{1}{\lambda}) \in \Gamma(\mathcal{T})$.
3. If $\lambda \in S(\mathcal{T})$ then $\frac{1}{z - \lambda} \in S(T)$.
4. $S(T) \subset \sigma(T)$. 
Relation induced by a Canonical System on $L^2(H, \mathbb{R}_+)$

Consider the maximal relation $\mathcal{R}$ on $L^2(H, \mathbb{R}_+)$ given by

$$\mathcal{R} = \{(f, g) \in (L^2(H, \mathbb{R}_+))^2 : f \in AC, Jf' = Hg\}.$$

The adjoint relation $\mathcal{R}_0 = R^*$, called as minimal relation is defined by

$$\mathcal{R}_0 = \{(f, g) \in (L^2(H, \mathbb{R}_+))^2 : \langle g, h \rangle = \langle f, k \rangle \text{ for all } (h, k) \in \mathcal{R}\}.$$

The minimal relation $\mathcal{R}_0$ is symmetric: $\mathcal{R}_0 \subset \mathcal{R}_0^* = \mathcal{R}$ and is given by

$$\mathcal{R}_0 = \{(f, g) \in \mathcal{R} : f(0+) = 0, \lim_{x \to \infty} f^*(x)Jh(x) = 0, (h, k) \in \mathcal{R}\}.$$
\( \beta(\mathcal{R}_0) \) is equal to the number of linearly independent solutions of the system 1 of whose class lie in \( L^2(H, \mathbb{R}_+) \).

It follows that \( \mathcal{R}_0 \) has equal defect indices, by Theorem 2 it has a self-adjoint extension say \( \mathcal{T} \).

Note: the limit circle case of the system 1. That implies for any \( z \in \mathbb{C}^+ \) the deficiency indices of \( \mathcal{R}_0 \) are \((2, 2)\). Suppose \( p \in D(\mathcal{R}) \setminus D(\mathcal{R}_0) \) such that \( \lim_{x \to \infty} p(x)^* Jp(x) = 0 \). Then the relation

\[
\mathcal{T}^{\alpha,p} = \{ (f, g) \in \mathcal{R} : f_1(0) \sin \alpha + f_2(0, z) \cos \alpha = 0 \}
\]

defines a self-adjoint relation.
We next discuss the spectrum of $\mathcal{T}^{\alpha,p}$. Let $u(x, z)$ and $v(x, z)$ be two linearly independent solutions of the system 1 with
\[
u(0, z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v(0, z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

Let $z \in \mathbb{C}^+$ and write $f(x, z) = u(x, z) + m(z)v(x, z) \in L^2(H, \mathbb{R}_+)$ satisfying $\lim_{x \to \infty} f(x, z)^*JP(x) = 0$. Let $T(x, z) = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}$ and
\[
w_\alpha(x, z) = \frac{1}{\sin \alpha + m(z) \cos \alpha} T(x, z) \begin{pmatrix} \cos \alpha \\ - \sin \alpha \end{pmatrix}.
\]
Let $z \in \rho(\mathcal{T}^{\alpha,p})$ then the resolvent operator $(\mathcal{T}^{\alpha,p} - z)^{-1}$ is given by

$$(\mathcal{T}^{\alpha,p} - z)^{-1} h(x) = \int_0^\infty G(x, t, z) H(t) h(t) dt$$

where $G(x, t, z) = \begin{cases} f(x, z) w_\alpha(t, \bar{z})^* & \text{if } 0 < t \leq x \\ w_\alpha(t, \bar{z}) f(x, \bar{z}) & \text{if } x < t \leq \infty \end{cases}$

This is unitarily equivalent with the integral operator (Hilbert Schmidt) $\mathcal{L}$ on $L^2(I, \mathbb{R}_+)$ given by

$$(\mathcal{L}g)(x) = \int_0^\infty L(x, t) g(t) dt, \quad L(x, t) = H^{\frac{1}{2}}(x) G(x, t, z) H^{\frac{1}{2}}(t).$$

Hence it has only discrete spectrum consisting of eigenvalues and possibly zero. By Theorem 3, $\mathcal{T}^{\alpha,p}$ has discrete spectrum consisting of eigenvalues.
Theorem 4

*The defect index* $\beta(R_0, z) = \dim R_{0z} = \dim N(R, \bar{z})$ of $R_0$ is constant on $\mathbb{C}$.

Proof.

Since $R_0$ is a symmetric relation, by Theorem 1 the defect index $\beta(R_0, z)$ is constant on upper and lower half planes. Suppose $\beta(R_0, \lambda) < 2$ for some $\lambda \in \mathbb{R}$. Since $\Gamma(R_0)$ is open, $\lambda \notin \Gamma(R_0)$ and hence $\lambda \in S(R_0)$. Since for each $\alpha \in (0, \pi]$, $T^{\alpha,p}$ is self-adjoint extension of $R_0$, $\lambda \in S(T^{\alpha,p}) = \sigma(T^{\alpha,p})$. Since $\sigma(T^{\alpha,p})$ consists of only eigenvalues, $\lambda$ is an eigenvalue for all boundary conditions $\alpha$ at 0. However, this is impossible unless $\beta(R_0, \lambda) = 2$. This completes the proof.
Theorem 5

Consider the canonical system 1 with \( \text{trace} H \equiv 1 \) then it prevails limit point case.

Proof.

Suppose it prevails the limit circle case. That means all solutions of 1 are in \( L^2(H, \mathbb{R}_+) \). By Theorem 4, for \( 0 \in \mathbb{R} \), \( \dim N(\mathcal{R}, 0) = 2 \). In particular, 0 is an eigenvalue and \( u(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( v(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) are the eigenfunctions of the relation \( \mathcal{R} \) in \( L^2(H, \mathbb{R}_+) \). However,

\[
\int_{0}^{\infty} u(x)^* H(x) u(x) dx + \int_{0}^{\infty} v(x)^* H(x) v(x) dx = \\
\int_{0}^{\infty} \text{trace} H(x) dx = \infty.
\]

This is a contradiction. It follows that the canonical system 1 has limit point case.
I.S. Kac.
On the Hilbert spaces, generated by monotone Hermitian matrix functions.
*Kharkov, Zap Mat. o-va, 22: 95–113, 1950*

Seppo Hassi, Henk De Snoo, and Henrik Winkler.
Boundary-value problems for two-dimensional canonical systems.

Joachim Weidmann.
Linear Operators in Hilbert Spaces.
*Springer-Verlag*, 1980

Remling, Christian.
Schrödinger operators and de Branges spaces.
Keshav Acharya, and Christian Remling.
Absolutely Continuous Spectrum of a Canonical System (In preparation).
Thank You!