On the classification of gapped quantum phases

Sven Bachmann[1]

[1]Department Mathematics
University of California, Davis

Arizona school of analysis and mathematical physics,
University of Arizona, Tucson, March 2012

Quantum spin systems

- A countable collection Γ (lattice) of finite dimensional quantum systems (spins)

$$\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{H}_x\,, \qquad \Lambda \subset \Gamma\,, \text{finite}$$

- The Heisenberg dynamics is generated by a local Hamiltonian: a sum of short range interactions $\Phi(X)$ with support in $X \subset \Lambda$

$$H_\Lambda = \sum_{X\subset\Lambda} \Phi(X)$$

- Ground states: Eigenstates ψ_Λ to the lowest eigenvalue (maybe group of low lying eigenvalues)
- The system is gapped if there is a lower bound on the spectral gap above the ground state energy, uniform in Λ.

‘Topological’ ground states

A jungle of ground state behaviours:

- ▷ Local order
- ▷ Symmetry breaking or not, unique vs multiple ground states
- ▷ Exponential/polynomial decay of correlations or no decay

All can be observed and categorized locally

Also: ‘topological order’ and edge states. The ground states are locally indistinguishable, but differ globally: depending on the underlying topology of the system, or at the edges.
E.g. Kitaev’s ‘toric code model’ defined on a 2-dimensional surface of genus g has 4^g degenerate ground states.

- ▷ Local disorder, sometimes called ‘topological order’

Note: A phase here is a family of models

Classification – gapped phases

- The question: what defines a gapped ground state phase?
- Models within a phase should be qualitatively equivalent, for example: same ground state degeneracy
- A conjecture in 1-d: all translation invariant chains with a unique gapped thermodynamic ground state are in the same phase, in particular simple product states
- Consensus: There cannot be a phase transition without closing the gap
- A definition: Two gapped systems are equivalent if there exists a smooth path of gapped Hamiltonians interpolating between them
- Also: The ground state spaces $\mathcal{S}$ within a phase should be mapped onto each other by local unitary transformations

Automorphic equivalence

Theorem.
Given a smooth path of uniformly gapped Hamiltonians $H(s)$ there is a cocycle of automorphisms $\alpha_{s,s'}$ of the algebra of observables s.t.

$$\mathcal{S}(s) = \mathcal{S}(s') \circ \alpha_{s,s'}$$

The maps $\alpha_{s,s'}$ are generated by a time dependent interaction $\Psi(X,s)$, which decays almost exponentially; $\alpha_{s,s'}$ extend to infinite systems

Concretely, the action of the quasi-local transformations $\alpha_s = \alpha_{s,0}$ on observables is given by

$$\alpha_s(A) = \lim_{n\to\infty} V_n^*(s) A V_n(s)$$

where $V_n(0) = 1$ and $V_n(s)$ solves a Schrödinger equation:

$$\frac{d}{ds} V_n(s) = \mathrm{i} D_n(s) V_n(s), \qquad \text{with} \qquad D_n(s) = \sum_{X\subset\Lambda_n} \Psi(X,s)$$

Product vacua with boundary states

- Now: $\Lambda \subset \mathbb{Z}$ and $\mathcal{H} = \mathbb{C}^{n+1}$
- We interpret the states as n distinguishable particles labeled $1, \ldots, n$, and a vacuum 0
- The Hamiltonian has nearest-neighbor interaction (hopping)

$$\Phi = \sum_{1 \le j \le n} |\hat{\phi}_{jj}\rangle\langle\hat{\phi}_{jj}| + \sum_{0 \le j < k \le n} |\hat{\phi}_{jk}\rangle\langle\hat{\phi}_{jk}|,$$

with, for $j \neq k = 0, \ldots, n$,

$$\phi_{jk} = |j, k\rangle - e^{-\mathrm{i}\theta_{kj}} \lambda_j^{-1} \lambda_k \, |k, j\rangle \qquad \phi_{jj} = |j, j\rangle$$

The parameters satisfy $0 < \lambda_j \neq 1$ for $j = 1, \ldots, n$, $\lambda_0 = 1$, and $\theta_{jk} = -\theta_{kj} \in \mathbb{R}$

PVBS ground states

- Each particle can appear at most once in the ground state
- The n_L particles that have $\lambda_i < 1$ are bound to the left edge, the n_R particles that have $\lambda_i > 1$ are bound to the right edge.
- Finite chains: 2^n ground states
- Thermodynamic limit: All converge to the product vacuum
- Half infinite chains: Edge states , 2^{n_L} on the right infinite chain

If $\lambda_i \neq 1$ for $i > 0$, the spectral gap if uniformly bounded below. Moreover, the exact gap in the thermodynamic limit γ satisfies

$$\gamma < \min_{i=1,\ldots,n} \left\{ 1 - \frac{2}{\lambda_i + \lambda_i^{-1}} \right\}$$

Note: the gap closes whenever $\lambda_i = 1$ for some index i.

Classification of phases

Lemma.

Two PVBS models belong to the same gapped phase (equivalence class) if and only if they have the same n_L and n_R.

- If $n_L \neq n_R$, the dimensions of the ground state spaces don't match, so they cannot be related by an automorphism
- Smoothly interpolating between the λ values yields a path of gapped Hamiltonians, as long as no λ crosses 1.

As an example of the use of the classes, we identify the class of a $SU(2)$-invariant spin-1 antiferromagnet, the AKLT model

Theorem.

The AKLT model belongs to the PVBS phase with $n_L = n_R = 1$.

- The thermodynamic phase is equivalent to a product state

Concluding remarks

What has been done:

- Two models belong to the same phase if for a given set of lattices Γ's, there is a local automorphism α_Γ of the observable algebra relating the ground state spaces.
- In particular, this takes edges and topologies into account
- In one dimension, the PVBS are simple representatives of classes with a unique bulk ground state
- They allow for a refined version of the product phase conjecture

Much more to do:

- In 1-d: More general PVBS, with multiple bulk ground states
- Higher dimensions, real topological phases
- Quantum phase transitions and universality