Schrödinger operators with decaying oscillatory potentials

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We investigate half-line Schrödinger operators $H = -\Delta + V$

0 will be a regular endpoint (and most of our statements will be independent of boundary condition)

$V$ should obey an oscillation condition and decay at $+\infty$

What can we say about the spectrum of $H$?

(Corollary of Weyl’s theorem for relatively compact perturbations)
If $V$ decays at infinity, in the sense that

$$\lim_{n \to \infty} \int_{n}^{n+1} |V(x)| \, dx = 0$$

then $\sigma_{\text{ess}}(H) = [0, +\infty)$

What about the decomposition into absolutely continuous, singular continuous, pure point spectrum? How stable is a.c. spectrum?
Decaying potentials with harmonic oscillations

- (the Wigner–von Neumann potential)
  Explicit potential on $(0, +\infty)$ with asymptotic behavior

\[ V(x) = -8 \frac{\sin 2x}{x} + O(x^{-2}), \quad x \to \infty \]

such that $-\Delta + V$ has eigenvalue $+1$ embedded in the a.c. spectrum $(0, +\infty)$

- (Atkinson, Harris–Lutz, Ben Artzi–Devinatz)
  If

\[ V(x) = \sum_{k=1}^{K} \lambda_k \frac{\sin(\alpha_k x)}{x^{\gamma_k}} + W(x) \]

with $\gamma_k > \frac{1}{2}$ and $W \in L^1$, then $H$ has purely a.c. spectrum on $(0, +\infty) \setminus \left\{ \frac{\alpha_k^2}{4} \mid 1 \leq k \leq K \right\}$

- (Weidmann)
  If $V$ has bounded variation and $V(x) \to 0$ as $x \to +\infty$, then $-\Delta + V$ has purely a.c. spectrum on $(0, +\infty)$
Generalized bounded variation: definition

- A function $\beta(x)$ has rotated bounded variation with phase $\phi$ if $e^{i\phi x} \beta(x)$ has bounded variation.
- A function $V(x)$ has generalized bounded variation with phases $\phi_1, \ldots, \phi_L$ ($L < \infty$) if

$$V(x) = \sum_{l=1}^{L} \beta_l(x) + W(x)$$

where $\beta_l$ has rotated bounded variation with phase $\phi_l$ and $W \in L^1$.
- Example of rotated bounded variation:

$$\frac{e^{-i(\phi x + \alpha)}}{(1 + x)\gamma}, \quad \text{with } \gamma > 0$$

- Example of generalized bounded variation:

$$\frac{\cos(\phi x + \alpha)}{(1 + x)\gamma}, \quad \text{with } \gamma > 0$$

or a linear combination of such terms.
Theorem 1. Let \( V : (0, \infty) \to \mathbb{R} \) be such that
- \( V \) has generalized bounded variation with set of phases \( A = \{\phi_1, \ldots, \phi_L\} \), i.e.

\[
V(x) = \sum_{l=1}^{L} \beta_l(x) + W(x)
\]

where \( e^{i\phi_l x} \beta_l(x) \) has bounded variation and \( W \in L^1 \)
- \( V \in L^1 + L^p \) for some \( p < \infty \)

Then the operator \( H = -\Delta + V \) on \( L^2(0, +\infty) \) satisfies
- \( \sigma_{ac}(H) = [0, +\infty) \)
- \( \sigma_{sc}(H) = \emptyset \)
- \( \sigma_{pp}(H) \cap (0, \infty) \subset \left\{ \frac{\eta^2}{4} \mid \eta \in \bigcup_{k=1}^{p-1} (A + \cdots + A) \right\} \) is a finite set

Application: slowly decaying Wigner–von Neumann type potentials

\[
V(x) = \sum_{k=1}^{K} \lambda_k \frac{\cos(\alpha_k x + \xi_k)}{x^{\gamma_k}} + W(x), \quad \gamma_k > 0, \quad W \in L^1
\]
Existence of embedded eigenvalues

**Theorem 2.** For a generic choice of \(\alpha_1, \ldots, \alpha_K\) and
\[
E = \frac{1}{4} (\pm \alpha_{j_1} \pm \cdots \pm \alpha_{j_{p-1}})^2,
\]
there exists an \(L^p\) potential of generalized bounded variation
\[
V(x) = \sum_{k=1}^{K} \lambda_k \frac{1}{x^\gamma} \cos(\alpha_k x + \xi_k(x)) + \beta_0(x), \quad x \geq x_0
\]
such that \(-\Delta + V\) has a real-valued eigenfunction \(u(x)\) at energy \(E\) with asymptotics
\[
\frac{1}{\sqrt{E}} u'(x) + iu(x) = Af(x)e^{i[\sqrt{E}x + \theta\infty]}(1 + o(1)), \quad x \to \infty
\]
with
\[
f(x) = \begin{cases} 
  x^{-C\lambda_{j_1} \ldots \lambda_{j_{p-1}}} & \gamma = \frac{1}{p-1} \\
  \exp \left(-\frac{C}{1-(p-1)\gamma} \lambda_{j_1} \ldots \lambda_{j_{p-1}} x^{1-(p-1)\gamma}\right) & \gamma \in \left(\frac{1}{p}, \frac{1}{p-1}\right)
\end{cases}
\]
and \(A, C > 0\)
Our Theorem 1 applies, in particular, to potentials of the form

$$V(x) = \tau(x) \sum_{k=1}^{K} c_k e^{i\phi_k x}$$

where $K < \infty$ and $\tau(x) \in L^p$ has bounded variation.

Can we generalize to potentials of the form

$$V(x) = \tau(x) W(x)$$

where $W(x)$ is a more general oscillatory function?
Theorem 3. Let \( V(x) = \tau(x)W(x) \), where

- \( \tau(x) \in L^p \) has bounded variation
- \( W(x) \) is periodic of period \( T \), such that
  - the Fourier series of \( W \) converges in \( L^1(0, T) \) to \( W \)
  - the Fourier coefficients of \( W \) obey \( \sum_{n \in \mathbb{Z} \setminus \{0\}} |\hat{W}_n| < \infty \) (for instance, let \( W \in L^2(0, T) \))

Then

- \( \sigma_{ac}(H) = [0, +\infty) \)
- \( \sigma_{sc}(H) = \emptyset \)
- \( \sigma_{pp}(H) \cap (0, \infty) \subset \left\{ \frac{k^2 \pi^2}{T^2} \mid k \in \mathbb{Z} \right\} \) is at most countable
Almost periodic potential × decay

**Theorem 4.** Let $V(x) = \tau(x) W(x)$, where

- $\tau(x) \in L^p$ has bounded variation
- $W(x) = \sum_{k=1}^{\infty} c_k e^{i\phi_k x}$, with $\sum |c_k|^\alpha < \infty$ for some $\alpha \in [0, 1)$

Then there is a set $S$, independent of boundary condition at 0, which supports the singular part of the spectral measure, such that

$$\dim_H S \leq (p - 1)\alpha$$

($\dim_H$ stands for Hausdorff dimension) and $\sigma_{ac}(H) = [0, +\infty)$.
All the theorems stated have analogs for orthogonal polynomials on the real line and unit circle.

Thank you for your attention!