A Gordon type theorem for measure perturbed Schrödinger operators

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Motivation

Three principles in modelling quasicrystals:
- locally regular/ordered
- globally aperiodic
- randomization

1976 Gordon: \( H = -\frac{d^2}{dx^2} + V \) does not have eigenvalues for “nice” \( V \in L_\infty(\mathbb{R}) \)

2000 Damanik/Stolz: same result for \( V \in L_{1,\text{loc}}(\mathbb{R}) \)

2011 S: generalization for measure perturbations
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Gordon measures

Definition

Let \( \mu = \mu_+ - \mu_- \) be a signed Borel measure. 
\( \mu \) uniformly locally bounded \( (\mu \in \mathcal{M}_{\text{loc,unif}}(\mathbb{R})) \)

\[ \iff \|\mu\|_{\text{loc}} := \sup_{x \in \mathbb{R}} |\mu| ([x, x + 1]) < \infty. \]

\( \mu \) Gordon measure \( \iff \mu \in \mathcal{M}_{\text{loc,unif}}(\mathbb{R}) \) and there exists \( (\mu^m) \) in \( \mathcal{M}_{\text{loc,unif}}(\mathbb{R}) \) of periodic measures with period sequence \( (p_m) \): \( p_m \to \infty \) and

\[ \forall C \geq 0 : \lim_{m \to \infty} e^{Cp_m} |\mu - \mu^m| ([-p_m, 2p_m]) = 0. \]
Definition of \( H_{\mu} \): form methods

\( \mu \in \mathcal{M}_{\text{loc,unif}}(\mathbb{R}) \implies \mu \) infinitesimally form small with respect to classical Dirichlet form

\[
D(\tau_0) := W^1_2(\mathbb{R}), \quad \tau_0(u, v) := \int u' \overline{v}',
\]

i.e.

\[
\forall \gamma \in (0, 1) \exists C_{\gamma} \geq 0 : \int |u|^2 \, d\mu \leq \gamma \tau_0(u, u) + C_{\gamma} \|u\|_{L^2(\mathbb{R})}^2 \quad (u \in D(\tau_0)).
\]

Hence:

\[
D(\tau_{\mu}) := W^1_2(\mathbb{R}), \quad \tau_{\mu}(u, v) := \int u' \overline{v}' + \int u \overline{v} \, d\mu
\]

densely defined, symmetric, semibounded from below and closed. Let \( H_{\mu} \sim \tau_{\mu} \), i.e.

\[
(H_{\mu} u \mid v) = \tau_{\mu}(u, v) \quad (u \in D(H_{\mu}), v \in D(\tau_{\mu})).
\]
Theorem (S 2011)

Let $\mu$ be a Gordon measure. Then $H_\mu$ does not have any eigenvalues.
Proof

Fix normalized initial condition at 0. Let \( u \) be the solution of \( H_\mu u = Eu \), \( u_m \) the solution of \( H_\mu^m u_m = Eu_m \) \( (m \in \mathbb{N}) \). By Gronwall inequality:

\[
\left\| \begin{pmatrix} u(x) \\ u'(x+) \end{pmatrix} - \begin{pmatrix} u_m(x) \\ u'_m(x+) \end{pmatrix} \right\| \leq Ce^{C|x|} |\mu - \mu^m| ([0, x])
\]

\[
\leq \frac{1}{4} \quad (x \in [-p_m, 2p_m], \ m \text{ large}).
\]

For solutions \( v \) to periodic measures with period \( p \):

\[
\max \left\{ \left\| \begin{pmatrix} v(-p) \\ v'(-p+) \end{pmatrix} \right\|, \left\| \begin{pmatrix} v(p) \\ v'(p+) \end{pmatrix} \right\|, \left\| \begin{pmatrix} v(2p) \\ v'(2p+) \end{pmatrix} \right\| \right\} \geq \frac{1}{2}.
\]

Hence,

\[
\limsup_{|x| \to \infty} \left( |u(x)|^2 + |u'(x+)|^2 \right) \geq \left( \frac{1}{4} \right)^2 > 0.
\]

Therefore, \( u \notin D(H_\mu) \).
Proof

Fix normalized initial condition at 0. Let $u$ be the solution of $H_\mu u = Eu$, $u_m$ the solution of $H_\mu^m u_m = Eu_m \ (m \in \mathbb{N})$. By Gronwall inequality:

$$\left\| \begin{pmatrix} u(x) \\ u'(x+) \end{pmatrix} - \begin{pmatrix} u_m(x) \\ u'_m(x+) \end{pmatrix} \right\| \leq C e^{C|x|} |\mu - \mu^m| (0, x)$$

$$\leq \frac{1}{4} \quad (x \in [-p_m, 2p_m], \ m \text{ large}).$$

For solutions $v$ to periodic measures with period $p$:

$$\max \left\{ \left\| \begin{pmatrix} v(-p) \\ v'(-p+) \end{pmatrix} \right\|, \left\| \begin{pmatrix} v(p) \\ v'(p+) \end{pmatrix} \right\|, \left\| \begin{pmatrix} v(2p) \\ v'(2p+) \end{pmatrix} \right\| \right\} \geq \frac{1}{2}.$$

Hence,

$$\limsup_{|x| \to \infty} \left( |u(x)|^2 + |u'(x+)|^2 \right) \geq \left( \frac{1}{4} \right)^2 > 0.$$

Therefore, $u \notin D(H_\mu)$. 