Ratio Asymptotics for General Orthogonal Polynomials

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March 13, 2012

[1] This material is based upon work supported by the National Science Foundation Graduate Research Fellowship under Grant No. 1144469.

Orthogonal Polynomials

- Let μ be a measure with compact and infinite support in $\mathbb{C}$.
- By performing Gram-Schmidt orthogonalization to $\{1, z, z^2, z^3, \ldots\}$, we arrive at the sequence of orthonormal polynomials $\{p_n(z;\mu)\}_{n\geq 0}$ satisfying

$$\int_{\mathbb{C}} p_n(z;\mu)\overline{p_m(z;\mu)}d\mu(z) = \delta_{nm}.$$

- The leading coefficient of p_n is $\kappa_n = \kappa_n(\mu)$ and satisfies $\kappa_n > 0$.

Orthogonal Polynomials (cont.)

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- Let $\mathrm{ch}(\mu)$ denote the convex hull of $\mathrm{supp}(\mu)$.
- For a set X, its polynomial convex hull is denoted $\mathrm{Pch}(X)$ and is defined by

$$\mathrm{Pch}(X) = \bigcap_{\text{polynomials } p \neq 0} \left\{ z : |p(z)| \leq \|p\|_{L^\infty(X)} \right\}.$$

- If $\overline{\mathbb{C}} \setminus \mathrm{Pch}(\mathrm{supp}(\mu))$ is simply connected, let $\phi : \overline{\mathbb{C}} \setminus \mathrm{Pch}(\mathrm{supp}(\mu)) \to \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ be the conformal map satisfying $\phi(\infty) = \infty$ and $\phi'(\infty) > 0$.

Monic Orthogonal Polynomials

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- $P_n(\cdot;\mu)$ satisfies

$$\|P_n(\cdot;\mu)\|_{L^2(\mu)} = \inf\{\|Q\|_{L^2(\mu)} : Q = z^n + \text{ lower order terms}\},$$

a property we call the *extremal property*.

Regularity and Root Asymptotics

- A measure is called *regular* if

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- How much more do we need to assume about μ to conclude

$$\lim_{n\to\infty} \frac{p_n(z;\mu)}{p_{n-1}(z;\mu)} = \phi(z) \qquad , \qquad z \notin \mathrm{ch}(\mu)?$$

Ratio Asymptotics

- If M is the multiplication by variable operator on $\overline{\mathrm{span}\{1, z, z^2, \ldots\}} \subseteq L^2(\mathbb{C}, \mu)$ and R_m is the projection onto the space of polynomials having degree at most m, then

Fact

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Fact

$$P_{n+1}(z; \mu) = \det(z - R_n M R_n).$$

- If we use $\{p_n(\cdot; \mu)\}_{n \geq 0}$ as an orthonormal basis then Cramer's rule yields:

$$\frac{P_{n-1}(z; \mu)}{P_n(z; \mu)} = (z - R_{n-1} M R_{n-1})^{-1}_{n,n}.$$

OPUC and OPRL

- Recall that for OPRL, there are sequences $\{a_n, b_n\}_{n\in\mathbb{N}}$ with $a_n > 0$ and $b_n \in \mathbb{R}$ so that

$$xp_n(x;\mu) = a_{n+1}p_{n+1}(x;\mu) + b_{n+1}p_n(x;\mu) + a_n p_{n-1}(x;\mu).$$

- For OPUC, there is a sequence $\{\alpha_n\}_{n\geq 0}$ with $\alpha_n \in \mathbb{D}$ so that

$$P_{n+1}(z;\mu) = zP_n(z;\mu) - \overline{\alpha}_n P_n^*(z;\mu)$$

where $P_n^*(z;\mu) = z^n\overline{P_n(1/\bar{z};\mu)}$.

Ratio Asymptotics for OPUC and OPRL

- From a 2004 result of Simon, one can deduce the following:

Theorem (Simon, 2004)

Suppose μ has compact support in the real line and let $\{a_n, b_n\}_{n\in\mathbb{N}}$ be the recursion coefficients for the orthonormal polynomials. Suppose $\mathcal{N} \subseteq \mathbb{N}$ is a subsequence so that for every $m \in \mathbb{Z}$ it holds that

$$\lim_{\substack{n\to\infty \\ n\in\mathcal{N}}} a_{n+m} = 1 \qquad , \qquad \lim_{\substack{n\to\infty \\ n\in\mathcal{N}}} b_{n+m} = 0.$$

Then

$$\lim_{\substack{n\to\infty \\ n\in\mathcal{N}}} \frac{p_n(z;\mu)}{p_{n-1}(z;\mu)} = \frac{z+\sqrt{z^2-4}}{2} \qquad , \qquad z \notin supp(\mu).$$

Ratio Asymptotics for OPUC and OPRL (cont.)

- An even easier argument yields:

Theorem

Suppose μ has support in the unit circle and let $\{\alpha_n\}_{n\geq 0}$ be the recursion coefficients. Suppose $\mathcal{N} \subseteq \mathbb{N}$ is a subsequence so that for every $m \in \mathbb{Z}$ it holds that

$$\lim_{\substack{n\to\infty \\ n\in\mathcal{N}}} \alpha_{n+m} = 0.$$

Then

$$\lim_{\substack{n\to\infty \\ n\in\mathcal{N}}} \frac{p_n(z;\mu)}{p_{n-1}(z;\mu)} = z \qquad , \qquad |z| > 1.$$

Ratio Asymptotics for the Unit Disk

- Our new result is the following:

Theorem (S., 2012)

Suppose μ *has support in the closed unit disk and suppose* $\mathcal{N} \subseteq \mathbb{N}$ *is a subsequence so that*

$$\lim_{\substack{n\to\infty \\ n\in\mathcal{N}}} \kappa_n \kappa_{n-1}^{-1} = 1.$$

Then

$$\lim_{\substack{n\to\infty \\ n\in\mathcal{N}}} \frac{p_n(z;\mu)}{p_{n-1}(z;\mu)} = z \qquad , \qquad |z| > 1$$

and the convergence is uniform on compact subsets of $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$.

Ratio Asymptotics for the Unit Disk (cont.)

Corollary (S., 2012)

Suppose μ has support in the closed unit disk and suppose

$$\lim_{n\to\infty} \kappa_n^{-1/n} = 1.$$

Then there exists a subsequence $\mathcal{N} \subseteq \mathbb{N}$ of asymptotic density 1 so that

$$\lim_{\substack{n\to\infty \\ n\in\mathcal{N}}} \frac{p_n(z;\mu)}{p_{n-1}(z;\mu)} = z \qquad , \qquad |z| > 1$$

and the convergence is uniform on compact subsets of $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$.

Relative Asymptotics

- Suppose we have two measures μ and ν.

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- What relationship between μ and ν is required to ensure

$$\lim_{n\to\infty} \frac{p_n(z;\mu)}{p_n(z;\nu)}$$

exists?

- For what values of z does the limit exist?

Example: Measures on the Circle

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Example: Measures on the Circle

- Let μ be arc-length measure on the unit circle so $p_n(z;\mu) = z^n$.
- Let $d\nu(z) = f(z)d\mu(z) + d\nu_{sing}(z)$
- If f is nice enough (i.e. $\log(f) \in L^1$), Szegő's Theorem on the unit circle implies

$$S(z) = \lim_{n\to\infty} \frac{p_n(z;\nu)}{z^n}$$

exists for all $z \notin \overline{\mathbb{D}}$ and provides us with an explicit form for S.

Uvarov Transform

- An example of a relative asymptotic result we can prove is::

Theorem (S., 2012)

Suppose μ *and* $x \in \mathbb{C}$ *satisfy*

$$\lim_{n\to\infty} \frac{|p_n(x;\mu)|^2}{\sum_{j=0}^{n-1} |p_j(x;\mu)|^2} = 0.$$

Then for any $t > 0$ *we have*

$$\lim_{n\to\infty} \frac{p_n(z;\mu + t\delta_x)}{p_n(z;\mu)} = 1 \qquad , \qquad z \notin ch(\mu)$$

and the convergence is uniform on compact subsets of $\overline{\mathbb{C}} \setminus ch(\mu)$.

Christoffel Transform

- For any $x \in \mathbb{C}$ we can define the *Christoffel Transform* of a measure μ as
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- Therefore, $(z - x)P_{n-1}(z;\nu^x)$ has the smallest $L^2(\mu)$ norm of all monic degree n polynomials *with a zero at* x.

Christoffel Transform (cont.)

- We can prove the following:

Theorem (S., 2012)

Let ν^x be the Christoffel Transform of μ and suppose

$$\lim_{n\to\infty} \frac{|p_n(x;\mu)|^2}{\sum_{j=0}^{n-1} |p_j(x;\mu)|^2} = 0.$$

Then

$$\lim_{\substack{n\to\infty \\ n\in\mathcal{N}}} \frac{(z-x)p_{n-1}(z;\nu^x)}{p_n(z;\mu)} = 1 \qquad , \qquad z \notin ch(\mu)$$

and the convergence is uniform on compact subsets of $\overline{\mathbb{C}} \setminus ch(\mu)$.

Key to the Proof

- As the preceding results indicate, the following fact is the key to proving those theorems:

Theorem (S., 2012)

For each $n \in \mathbb{N}$ choose a polynomial Q_n of degree exactly n and having leading coefficient τ_n so that

1. $\lim_{n\to\infty} \|Q_n\|_{L^2(\mu)} = 1$,
2. $\lim_{n\to\infty} \tau_n/\kappa_n(\mu) = 1$.

Then

$$\lim_{n\to\infty} \frac{Q_n(z)}{p_n(z;\mu)} = 1 \qquad , \qquad z \notin ch(\mu)$$

and the convergence is uniform on compact subsets of $\overline{\mathbb{C}} \setminus ch(\mu)$.

- In fact, we may send $n \to \infty$ through a subsequence.

Saff's Formula

- The main ingredient in the proof of the key Theorem is the following formula due to Saff:

Proposition (Saff, 2010)

Let Q be a polynomial of degree at most n and suppose $p_n(z;\mu) \neq 0$. Then

$$\frac{Q(z)}{p_n(z;\mu)} = \frac{\int \frac{\overline{p_n(w;\mu)}Q(w)}{z-w} d\mu(w)}{\int \frac{|p_n(w;\mu)|^2}{z-w} d\mu(w)}$$

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$$= 1 + \frac{\int \frac{\overline{p_n(w;\mu)}(Q(w)-p_n(w;\mu))}{z-w} d\mu(w)}{\int \frac{|p_n(w;\mu)|^2}{z-w} d\mu(w)}$$

Saff's formula (cont.)

- Now all it takes is the Cauchy-Schwartz inequality:

$$\left| \int \frac{\overline{p_n(w;\mu)}(Q(w) - p_n(w;\mu))}{z - w} d\mu \right|^2 \leq C_K \| Q(\cdot) - p_n(\cdot;\mu)\|^2_{L^2(\mu)}$$

- We expand the norm as

$$\|Q(\cdot) - p_n(\cdot;\mu)\|^2_{L^2(\mu)} = \|Q\|^2_{L^2(\mu)} + \|p_n\|^2_{L^2(\mu)} - 2\mathrm{Re}\langle Q, p_n\rangle_\mu .$$

Summary

- Ratio asymptotic results are well understood on the unit circle and real line in terms of the recursion coefficients for the orthonormal polynomials.
- Using some new techniques, we can prove analogous results when no such recursion relation exists.
- These techniques also yield results about the stability of he orthonormal polynomials under certain perturbations of the measure.