Bounds on fluctuations for Mallows random permutations:
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March 14, 2012

- The length of the longest increasing subsequence of a random Mallows permutation. *J. Theoret. Probab.* 2011 (to appear) joint work with Carl Mueller, UR
- and joint work with Meg Walters, UR, in preparation.
What is a Mallows random permutation?

Given $q \in (0, \infty)$,

$$\mu_n(q) = q I_n(\pi) P_n(q),$$

where the number of inversions $I_n(\pi) = \sum 1 \leq i < j \leq n \{\pi_i > \pi_j\}$.

**Fact:** $P_n(q)$ is the "Poincaré polynomial".

$$P_n(q) = n \prod_{k=1}^{\infty} (1 - q^k)^{1-q^k} = \lfloor n \rfloor! \text{q-factorial}.$$
What is a Mallows random permutation?

\[ \pi = (3, 4, 6, 2, 5, 1) \]

Given \( q \in (0, \infty) \),

\[ \mu_{n,q}(\{\pi\}) = \frac{q^{l_n(\pi)}}{P_n(q)}, \]

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\[ l_n(\pi) = \sum_{1 \leq i < j \leq n} 1\{\pi_i > \pi_j\}. \]
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Can define a classical Hamiltonian on $S_n$:

$$H_n(\pi) = \frac{1}{n} I_n(\pi) = \frac{1}{n} \sum_{1 \leq i < j \leq n} 1\{\pi_i > \pi_j\}.$$ 

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3/18. *q*-Stirling formula

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$$\sim n! \ e^{nA(\beta)} B(\beta)$$

$$A(\beta) = \int_{0}^{1} \ln \left( \frac{1 - e^{-\beta x}}{\beta x} \right) \, dx, \quad B(\beta) = \sqrt{\frac{e^\beta - 1}{\beta}}.$$
4/18. A weak limit law

Example:

\[ \begin{align*}
\pi_1 &= 3 \\
\pi_2 &= 4 \\
\pi_3 &= 6 \\
\pi_4 &= 2 \\
\pi_5 &= 5 \\
\pi_6 &= 1
\end{align*} \]

Empirical measure on \([0, 1]^2\)

\[ \hat{\rho}_{n,\pi} = \frac{1}{n} \sum_{i=1}^{n} \delta(i/n, \pi_i/n) \]
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Theorem. For \(\beta \in \mathbb{R}\) fixed, take \(q_n(\beta) = \exp(-\beta/n)\).

There exists a density \(\rho_\beta\) on \([0, 1]^2\) such that, for any continuous function \(\varphi\) on \([0, 1]^2\),

\[
\mu_{n, q_n(\beta)} \left\{ \pi \in S_n : \left| \int \varphi \, d\hat{\rho}_{n, \pi} - \int \varphi \, d\rho_\beta \right| > \epsilon \right\} \to 0 \quad \text{as} \quad n \to \infty,
\]

for each fixed \(\epsilon > 0\).
Denote: \( \mathbf{x} = (x^1, x^2) \in [0, 1]^2 \).

Boltzmann-Gibbs measure on \(([0, 1]^2)^n\):

\[
d\mu_{n, \beta}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \frac{e^{-\beta H_n(\mathbf{x}_1, \ldots, \mathbf{x}_n)}}{Z_n(\beta)} \, d\mathbf{x}_1 \cdots d\mathbf{x}_n,
\]

\[
H_n(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \frac{1}{n} \sum_{1 \leq i < j \leq n} h(\mathbf{x}_i, \mathbf{x}_j),
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\[
h(\mathbf{x}_i, \mathbf{x}_j) = \mathbf{1}\{(x^1_i - x^1_j)(x^2_i - x^2_j) < 0\}.
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\[
h(\mathbf{x}_i, \mathbf{x}_j) = 1\{ (x_i^1 - x_j^1)(x_i^2 - x_j^2) < 0 \}.
\]

Then \( \rho_\beta \) is the unique measure on \([0, 1]^2\) satisfying

\[
\frac{d\rho_\beta(\mathbf{x})}{d\mathbf{x}} = \frac{1}{Z(\beta)} \exp \left( -\beta \int_{[0,1]^2} h(\mathbf{x}, \mathbf{x}') \, d\rho_\beta(\mathbf{x}') \right).
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\[
= \frac{(\beta/2) \sinh(\beta/2)}{\left( e^{\beta/4} \cosh\left( \frac{\beta}{2} [x - y] \right) - e^{-\beta/4} \cosh\left( \frac{\beta}{2} [x + y - 1] \right) \right)^2}.
\]
For $\pi \in S_n$,

$$L_n(\pi) = \max\{k \leq n : \exists i_1 < \cdots < i_k \text{ s.t. } \pi_{i_1} < \cdots < \pi_{i_k}\}.$$
6/18. Length of the Longest Increasing Subsequence

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For the uniform measure \( \mu_n \) on \( S_n \) (\( \beta = 0 \)),

\[
\lim_{n \to \infty} \mu_n \left\{ \pi : |n^{-1/2}L_n(\pi) - 2| > \epsilon \right\} = 0,
\]

for all \( \epsilon > 0 \).
7/18. Hammersley’s proof: \( n^{-1/2} \mathbb{E} L_n \) converges
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$\mathbb{E} L_{4n^2} \geq 2 \mathbb{E} L_{n^2}$
7/18. Hammersley’s proof: $n^{-1/2} \mathbb{E} L_n$ converges

Extend the definition of $L_n$ from permutations to point processes

$L(x_1, \ldots, x_n) = \max \{ k : \exists i_1 < \cdots < i_k \text{ s.t. } h(x_{i_j}, x_{i_\ell}) = 0, \forall j, \ell \leq k \}$

Also defined for random point processes.
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Also defined for random point processes.

\[
\mathbb{E}[L_{\text{Poisson}}((x+y)^2)] \geq \mathbb{E}[L_{\text{Poisson}}(x^2)] + \mathbb{E}[L_{\text{Poisson}}(y^2)]
\]

\[
\Rightarrow x^{-1} \mathbb{E}[L_{\text{Poisson}}(x^2)] \text{ converges by Fekete’s theorem.}
\]
Suppose $\rho$ is a measure on $[0, 1]^2$, satisfying

$$\exists C < \infty, \quad \frac{1}{C} \leq \frac{d\rho(x)}{dx} \leq C, \quad \forall x \in [0, 1]^2$$
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**Theorem.** (Deuschel, Zeitouni) Let $\rho^n = i.i.d.,$ product measure

$$\forall \epsilon > 0, \quad \lim_{n \to \infty} \rho^n(|n^{-1/2}L(x_1, \ldots, x_n) - I(\rho)| > \epsilon) = 0,$$
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I(\rho) = \max I(\rho, \gamma) \text{ over curves } \gamma: [0, 1] \to [0, 1]^2,
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\gamma^1(t), \gamma^2(t) \text{ non-decreasing}
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$$I(\rho, \gamma) = 2 \int_0^1 \left[ \frac{d\rho}{dx}(\gamma(t)) \frac{d\gamma^1}{dt} \cdot \frac{d\gamma^2}{dt} \right]^{1/2} dt.$$
8/18. Idea of proof and extension to Mallows

\[ I(\rho, \gamma) = 2 \int_0^1 \left[ \frac{d\rho}{d\gamma}(\gamma(t)) \frac{d\gamma_1}{dt} \cdot \frac{d\gamma_2}{dt} \right]^{1/2} \, dt \]
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**Thm.** (Mueller and S) Let \( q_n(\beta) = \exp(-\beta/n) \),

\[ \forall \epsilon > 0, \lim_{n \to \infty} \mu_{n,q_n(\beta)} \{|n^{-1/2}L_n(\pi) - \mathcal{L}(\beta)| > \epsilon\} = 0, \]

where

\[ \mathcal{L}(\beta) = 2\beta^{-1/2} \sinh^{-1}(\sqrt{e^\beta - 1}). \]
Weak conditional correlations

\[ -\frac{\beta}{n} \sum_{i<j} h(x_i, y_i; x_j, y_j) \]

\[ e^\alpha \]

\[ \text{Let boxes } = k^2. \]

\[ O(nk^2) \text{ points per box.} \]

\[ O(k) \text{ boxes in cross.} \]

\[ H_n \text{ has 1}_n \text{ factor.} \]

Exponential interaction for box \( O(1/k) \).

Shannon Starrr

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Let $U$ be Bernoulli-2/5 and $V$ be Bernoulli-5/6, independently.

If $V = 1$, let $X = Y = U$.

If $V = 0$, let $X = 1, Y = 0$. 
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For two random variables, can couple $X$ and $Y$ so that

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$$\mathbb{P}(X = Y) = 1 - \|\mu_X - \mu_Y\|_{TV}$$

$$\|\mu_X - \mu_Y\|_{TV} = \max_A |P(X \in A) - P(Y \in A)| = \frac{1}{2} \int |f_X(x) - f_Y(x)| \, dx$$
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$O\left(\frac{n}{k^2}\right)$ points per box.

$O(k)$ boxes in cross.

Exponential term $O(1/k)$ per particle.
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So fraction of points that are not coupled to IID: $O(1/k)$.
Conclusion of proof

- For the empirical measure $\hat{\rho}_{n,\pi} = \frac{1}{n} \sum_{i=1}^{n} \delta(i/n,\pi_i/n)$,
  
  \[
  \mu_{n,q_n(\beta)} \{ | \int_{[0,1]^2} \varphi \, d\hat{\rho}_{n,\pi} - \int_{[0,1]^2} \varphi \, d\rho_{\beta} | > \epsilon \} \to 0 \text{ for each continuous } \varphi \text{ and each } \epsilon > 0.
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- In particular for any finite number \( k^2 \) boxes, the point counts converge in probability.
For the empirical measure $\hat{\rho}_{n,\pi} = \frac{1}{n} \sum_{i=1}^{n} \delta(i/n, \pi_i/n)$,

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Because of the coupling, we can couple inside each box to the Deuschel-Zeitouni model with $\rho = \rho_\beta$ with $O(1/k)$ fraction of particle number fluctuation.
Conclusion of proof

- For the empirical measure $\hat{\rho}_{n,\pi} = \frac{1}{n} \sum_{i=1}^{n} \delta(i/n, \pi_i/n)$, $\mu_{n,q_n(\beta)} \{ | \int_{[0,1]^2} \varphi \, d\hat{\rho}_{n,\pi} - \int_{[0,1]^2} \varphi \, d\rho_{\beta} | > \epsilon \} \rightarrow 0$ for each continuous $\varphi$ and each $\epsilon > 0$.

- In particular for any finite number $k^2$ boxes, the point counts converge in probability.

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- Taking $k \rightarrow \infty$ after $n \rightarrow \infty$, and using monotonicity of $L$ show that one can reduce to the Deuschel-Zeitouni optimization problem.
Conclusion of proof

For the empirical measure $\hat{\rho}_{n,\pi} = \frac{1}{n} \sum_{i=1}^{n} \delta(i/n,\pi_i/n)$, 

$$\mu_{n,q_n(\beta)} \left\{ \left| \int_{[0,1]^2} \varphi \, d\hat{\rho}_{n,\pi} - \int_{[0,1]^2} \varphi \, d\rho_{\beta} \right| > \epsilon \right\} \to 0 \text{ for each continuous } \varphi \text{ and each } \epsilon > 0.$$ 

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Moreover, it is a calculus exercise to see that for $\rho = \rho_{\beta}$, $I(\rho, \gamma)$ is attained at $\gamma = \text{diagonal}$, and gives the formula

$$L(\beta) = 2\beta^{-1/2} \sinh^{-1}(\sqrt{e^\beta - 1}).$$
12/18. Conclusion of proof

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In particular for any finite number \( k^2 \) boxes, the point counts converge in probability.

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Moreover, it is a calculus exercise to see that for \( \rho = \rho_\beta \),
\[ \mathcal{I}(\rho, \gamma) \] is attained at \( \gamma = \text{diagonal} \), and gives the formula
\[ \mathcal{L}(\beta) = 2\beta^{-1/2} \sinh^{-1}(\sqrt{e^\beta - 1}) \, . \]

After a reparametrization \( \rho_\beta(x'(x), y'(y)) \propto (1 - \beta xy)^{-2} \).
13/18. Bounds on the fluctuations

Let \( \# \text{ boxes} = k^2 \).

\( O\left(\frac{n}{k^2}\right) \) points per box.

\( O(k) \) boxes in cross.

Coupling failure rate \( O(1/k) \).

\[ -\frac{\beta}{n} \sum_{i<j} h(x_i, y_i; x_j, y_j) \]

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\(k = O(n^{1/4})\)
14/18. Bounds on the counts

Four-square problem:

\[
P_q \left( \begin{array}{cc}
  n_{11} & n_{12} \\
  n_{21} & n_{22}
\end{array} \right) = P_1 \left( \begin{array}{cc}
  n_{11} & n_{12} \\
  n_{21} & n_{22}
\end{array} \right) \cdot W_q \left( \begin{array}{cc}
  n_{11} & n_{12} \\
  n_{21} & n_{22}
\end{array} \right)
\]

\[
q^{n_{12}n_{21}} \frac{(n_{11} + n_{12})!(n_{11} + n_{21})!(n_{12} + n_{22})!(n_{21} + n_{22})!}{(n_{11})!(n_{12})!(n_{21})!(n_{22})!(n_{11} + n_{12} + n_{21} + n_{22})!}
\]

where \( \{n\}! = [n]!/n! \).
Large deviations for 4-square

Stirling formula → relative entropy:

\[
\frac{1}{n} \ln \mathbb{P}_1 \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix} = \frac{1}{n} \ln \left( \frac{n!}{\prod_{i,j=1}^{2} n_{ij}!} \right) \to - \sum_{i,j=1}^{2} \rho_{ij} \ln \left( \frac{\rho_{ij}}{|\Lambda_{ij}|} \right)
\]

for \( n \to \infty \), with \( n_{ij}/n \to \rho_{ij} \), where \( |\Lambda_{ij}| = \text{area of sub-square } \Lambda_{ij} \).
Stirling formula $\rightarrow$ relative entropy:

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\underline{$q$-Stirling formula:}

\[
\frac{1}{n} \ln \{n\}!_{q=\exp(-\beta/n)} \rightarrow A(\beta) = \int_{0}^{1} \ln \left( \frac{1 - e^{-\beta x}}{\beta x} \right) \, dx .
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\]

\[
\frac{1}{n} \ln \{n_{ij}\}! \bigg|_{q=\exp(-\beta/n)} \rightarrow \rho_{ij} A(\beta \rho_{ij}).
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Large deviations for 4-square

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\[ e^{\beta \left( \begin{array}{cc} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{array} \right)} = \lim_{n \rightarrow \infty} \frac{1}{n} \ln \mathcal{W}_{q_n}(\beta) \left( \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix} \right) \]
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Ultimately, we get fluctuating particle number by coupling:

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We have to compare \( L_n \) in the Mallows measure to \( L_{N_n} \) in the Deuschel-Zeitouni model.

We can either settle for \( O(n^{(3/8)^+}) \) bounds, or we can prove \( O(n^{(1/4)^+}) \) bounds along subsequences.
All we need to do is show that the area on the right hand picture is $O(n^{-1/2})$: each “box” is $O(n^{-1})$ and there are $O(n^{1/2})$ “boxes.”
Aldous and Diaconis proved that on a horizontal slice, the length of the LIS behaves locally like a Poisson point process:

\[ \text{Intensity} \sqrt{\frac{1-y}{1-x}} \]

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\[\text{LIS to NE} \quad \Delta y \quad \text{LIS to SW} \quad \text{Intensity } \sqrt{\frac{y}{x}}\]

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Thanks for your attention!