Existence of Thermodynamic Limit for Interacting Quantum Particles in Random Media

Nikolaj Veniaminov

Laboratoire Analyse Géométrie et Applications, Université Paris 13

Arizona School of Analysis and Mathematical Physics
University of Arizona
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Outline

1 Model
   - Multiparticle Random Schrödinger Operator
   - Thermodynamic Limit and Problem We Study

2 Existence Theorems
   - Main Result: Energy Convergence
   - Convexity and Entropy Convergence

3 Further Advances: Poisson Pieces Model
System of Interacting Quantum Particles in Random Medium

- One-particle random Schrödinger operator $H_\omega$ on $\mathcal{H} = L^2(\mathbb{R}^d)$

Anderson model:

$$H_\omega = -\Delta + \sum_{j \in \mathbb{Z}^d} \omega_j v(\cdot - j), \quad \{\omega_j\}_{j \in \mathbb{Z}^d} \text{ iid}$$

$n$-particle operator:

$$H_\omega(n) = \sum_{i=1}^n H \otimes \cdots \otimes H \underbrace{\otimes}_{i \text{ times}} H \omega \otimes \underbrace{\otimes}_{n-i \text{ times}} H$$

on $\mathcal{H}_n = \bigotimes_n \mathcal{H} = L^2(\mathbb{R}^d)$

- for classical particles,
- $\bigwedge_n \mathcal{H} = L^2(\mathbb{R}^d)$
- for fermions,
- $\text{Sym}_n \mathcal{H} = L^2 + (\mathbb{R}^d)$
- for bosons
One-particle random Schrödinger operator $H_\omega$ on $\mathcal{H} = L^2(\mathbb{R}^d)$

Anderson model:

$$H_\omega = -\Delta_d + \sum_{j \in \mathbb{Z}^d} \omega_j \nu(\cdot - j), \quad \{\omega_j\}_{j \in \mathbb{Z}^d} \text{ iid}$$
System of Interacting Quantum Particles in Random Medium

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- Anderson model:
  $$H_\omega = -\Delta_d + \sum_{j \in \mathbb{Z}^d} \omega_j \psi(\cdot - j), \quad \{\omega_j\}_{j \in \mathbb{Z}^d} \text{ iid}$$

- $n$-particle operator:
  $$H_\omega(n) = \sum_{i=1}^{n} \underbrace{1_{s_j} \otimes \ldots \otimes 1_{s_j}}_{i - 1 \text{ times}} \otimes H_\omega \otimes \underbrace{1_{s_j} \otimes \ldots \otimes 1_{s_j}}_{n - i \text{ times}} + \sum_{1 \leq i < j \leq n} U(x^i - x^j)$$
System of Interacting Quantum Particles in Random Medium

- One-particle random Schrödinger operator $H_\omega$ on $\mathcal{H} = L^2(\mathbb{R}^d)$
- Anderson model:
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- $n$-particle operator:
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on
\[ \mathcal{H}^n = \begin{cases} \bigotimes^n \mathcal{H} = L^2(\mathbb{R}^{nd}), & \text{for classical particles,} \\ \bigwedge^n \mathcal{H} = L^2_-(\mathbb{R}^{nd}), & \text{for fermions,} \\ \text{Sym}^n \mathcal{H} = L^2_+ (\mathbb{R}^{nd}), & \text{for bosons} \end{cases} \]
Let $\Lambda \subset \mathbb{R}^d$. Thermodynamic limit:

\[ \Lambda \to \mathbb{R}^d, \; n \to +\infty, \; \text{so that} \; \frac{n}{|\Lambda|} \to \rho > 0 \]
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General question:

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Thermodynamic Limit and Problem We Study

- Let $\Lambda \subset \mathbb{R}^d$. Thermodynamic limit:
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- Let $H_\omega(\Lambda, n)$ be a restriction of $H_\omega(n)$ to $\Lambda \subset \mathbb{R}^d$.

- General question:
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  \]

Example

Let $E_\omega(\Lambda, n)$ be the ground state energy of $H_\omega(\Lambda, n)$. Question:

\[
E_\omega(\Lambda, n) \xrightarrow{\Lambda \to \mathbb{R}^d, n/|\Lambda| \to \rho} ?
\]
Entropy:

\[ S_\omega(E, \Lambda, n) = \log \left( \text{card}\{\text{eigenvalues of } H_\omega(\Lambda, n) \text{ less than } E\} \right) \]
Entropy and Internal Energy

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- Energy as a function of entropy: \( E_\omega(\Lambda, n, S) \)
Entropy and Internal Energy

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Energy as a function of entropy: \( E_\omega(\Lambda, n, S) \)

\( E_\omega(\Lambda, n, 0) \) is the ground state energy of \( H_\omega(\Lambda, n) \)
Existence of Thermodynamic Limit for Energy

Theorem (main)

Suppose:

$H_{\omega}$ is uniformly lower bounded: $H_{\omega} \geq -C$, $\forall \omega$,

$H_{\omega}$ satisfies a decorrelation at a distance estimate,

Interactions are stable: $H_{\omega}(\Lambda, n) \geq -Cn$, $\forall \Lambda$, $\forall \omega$,

Interactions are tempered: $\exists A, \lambda > d, R_0$ such that $|U(x)| \leq A|x|^{-\lambda}$ for $|x| > R_0$.

Then the energy admits thermodynamic limit:

$E_{\omega}(\Lambda, n, S) \overset{n \rightarrow \infty}{\longrightarrow} \varepsilon(\rho, \sigma)$,

$\Lambda \rightarrow \mathbb{R}^d$, $n |\Lambda| \rightarrow \rho$, $S \rightarrow \sigma \geq 0$.

The energy density $\varepsilon(\rho, \sigma)$ is a deterministic function.
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- \( H_\omega \) satisfies a decorrelation at a distance estimate,
- \( H_\omega \) satisfies a tempering estimate with constants \( A, \lambda > d, R_0 \) such that

  \[
  |U(x)| \leq A |x|^{-\lambda} \quad \text{for} |x| > R_0.
  \]

Then the energy admits thermodynamic limit:

\[
E_\omega(\Lambda, n, S) \xrightarrow{n \to \infty} \varepsilon(\rho, \sigma), \quad \Lambda \to \mathbb{R}^d, \quad n \to \infty.
\]

The energy density \( \varepsilon(\rho, \sigma) \) is a deterministic function.
Existence of Thermodynamic Limit for Energy

**Theorem (main)**

*Suppose:*

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- $H_\omega$ satisfies a decorrelation at a distance estimate,
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Then the energy admits thermodynamic limit: $E_\omega(\Lambda, n, S_n) \to \epsilon(\rho, \sigma), \Lambda \to \mathbb{R}^d, n|\Lambda| \to \rho, S_n \to \sigma \geq 0$.

The energy density $\epsilon(\rho, \sigma)$ is a deterministic function.
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Existence of Thermodynamic Limit for Energy

**Theorem (main)**

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- Interactions are tempered: $\exists A, \lambda > d, R_0$ such that

$$|U(x)| \leq A|x|^{-\lambda} \text{ for } |x| > R_0.$$

Then the energy admits thermodynamic limit:

$$\frac{E_\omega(\Lambda, n, S)}{n} \xrightarrow{L^2} \varepsilon(\rho, \sigma), \quad \Lambda \to \mathbb{R}^d, \quad \frac{n}{|\Lambda|} \to \rho, \quad \frac{S}{n} \to \sigma \geq 0.$$

The energy density $\varepsilon(\rho, \sigma)$ is a deterministic function.
Some Corollaries

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- $\varepsilon(\rho, \sigma)$ is convex in $\rho^{-1}$ and $\sigma$
- Entropy as a function of energy admits thermodynamic limit:

$$\frac{S_\omega(E, \Lambda, n)}{n} \to \sigma(\rho, \varepsilon), \quad \Lambda \to \mathbb{R}^d, \quad \frac{n}{|\Lambda|} \to \rho, \quad \frac{E}{n} \to \varepsilon \in \text{Ran } \varepsilon(\rho, \cdot),$$

where $\sigma(\rho, \cdot)$ is an inverse of $\varepsilon(\rho, \cdot)$.
What to Study Next?

One can study $\Psi(\Lambda, n)$, the ground state wavefunction of $H(\Lambda, n)$, in the thermodynamic limit.

Only fermionic case: $n$ bosons give $\sim n^2$ interactions, $n$ fermions give effectively $\sim n$ interactions.

Reference model for $d = 1$: domain $\Lambda$ becomes $[0, L]$.

Consider Poisson point process on $\mathbb{R}^3$: points inside $[0, L]$ define positions of walls.

$H(\omega) = -\Delta D$ with Dirichlet boundary conditions at these points.

Description of $\Psi(\Lambda, n)$ is obtained for small $\rho$.

In particular:

1. Loose functional subspace is found.
2. Slater determinant type structure.
3. Autocorrelation function and decorrelation length are estimated.
One can study $\Psi_\omega(\Lambda, n)$, the ground state wavefunction of $H_\omega(\Lambda, n)$, in the thermodynamic limit.
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The End

Thank you for your attention!