Non-equilibrium state of a leaking photon cavity pumped by a random atomic beam

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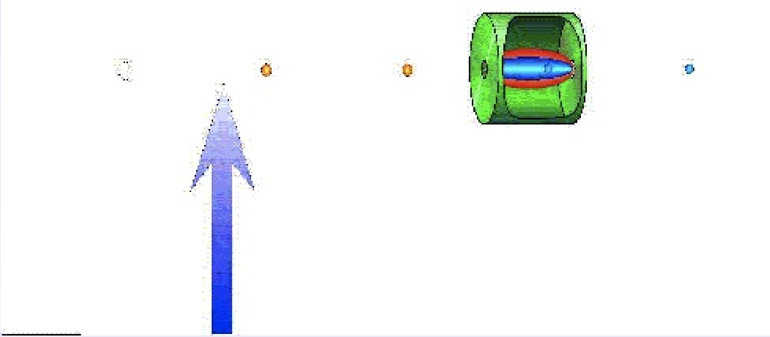


The picture is partially copied from http://www.mi.infm.it/a6/a6gp/prod02.htm

Introduction

The cavity is modeled by a harmonic oscillator on the Hilbert space $\mathcal{H}_C$. The atoms are described by a quantum spin chain on the Hilbert space $\mathcal{H}_A = \otimes_{n\geq 1}\mathcal{H}_{A_n}$.
The Hamiltonian of the cavity is

$$H_C = \epsilon b^* b \otimes \mathbb{1}$$

and the Hamiltonian for the n-th atom is

$$H_{A_n} = \mathbb{1} \otimes E a_n^* a_n.$$

The interaction between the n-th atom and the cavity depends on a time

$$W_n(t) = \chi_{[(n-1)\tau, n\tau]}(t)(\lambda a_n^* a_n \otimes (b^* + b)).$$

The Hamiltonian of the system is

$$H(t) = H_C + \sum_{n\geq 1}(H_{A_n} + W_n(t))$$
$$= \epsilon b^* b \otimes \mathbb{1} + \sum_{n\geq 1} \mathbb{1} \otimes E a_n^* a_n + \sum_{n\geq 1} \chi_{[(n-1)\tau, n\tau]}(t)(\lambda a_n^* a_n \otimes (b^* + b)).$$

When $t \in [(n-1)\tau, n\tau]$, only the n-th atom interacts with the cavity. Let us denote the Hamiltonian for the n-th atom and the cavity as

$$H_n = \epsilon b^* b \otimes \mathbb{1} + \mathbb{1} \otimes E a_n^* a_n + \lambda(b^* + b) \otimes a_n^* a_n.$$

In the case of the leaking cavity the system is described by both Hamiltonian and the dissipative part.
For any states ρ_C on $\mathcal{H}_C$ and ρ_A on $\mathcal{H}_A$ the generator of the dynamics is

$$L(t)(\rho_C \otimes \rho_A) = -i[H(t), \rho_C \otimes \rho_A] + \sigma b(\rho_C \otimes \rho_A) b^* - \frac{\sigma}{2}\{b^* b, \rho_C \otimes \rho_A\}.$$

So when $t \in [(n-1)\tau, n\tau]$ the generator of the dynamics is

$$L_n(\rho_C \otimes \rho_A) = -i[H_n, \rho_C \otimes \rho_A] + \sigma b(\rho_C \otimes \rho_A) b^* - \frac{\sigma}{2}\{b^* b, \rho_C \otimes \rho_A\}.$$

The state of the system $\rho_S(t)$ at time t is given by the Liouville's differential equation

$$\frac{d}{dt}\rho_S(t) = L(t)(\rho_S(t)).$$

Suppose the initial state of the system is

$$\rho_S = \rho_C \otimes \bigotimes_{k \geq 1} \rho_k,$$

where ρ_C is the initial state of the cavity and ρ_k is the initial state on $\mathcal{H}_{A_k}$ s.t. ρ_k commutes with $a_k^* a_k$, for example one could take $\rho_k = e^{-\beta E a_k^* a_k}/(1 + e^{-\beta E})$.

If $t = n\tau + \nu \in [n\tau, (n+1)\tau]$ the state at time t can be written as

$$\rho_S(t) = e^{\nu L_{n+1}} e^{\tau L_n} ... e^{\tau L_2} e^{\tau L_1} (\rho_C \otimes \bigotimes_{k=1}^{n+1} \rho_k).$$

The state of the cavity at time $t = n\tau$ is

$$\rho_C(t) = \rho_C^{(n)} = \mathrm{Tr}_{\mathcal{H}_A} \rho_S(t) = \mathrm{Tr}_{\mathcal{H}_A} [e^{\tau L_n} ... e^{\tau L_2} e^{\tau L_1} (\rho_C \otimes \bigotimes_{k=1}^{n} \rho_k)]$$

$$= \mathrm{Tr}_{\mathcal{H}_{A_n}} [e^{\tau L_n} (\mathrm{Tr}_{\mathcal{H}_{A_{n-1}}} ... \mathrm{Tr}_{\mathcal{H}_{A_1}} e^{\tau L_{n-1}} ... e^{\tau L_2} e^{\tau L_1} (\rho_C \otimes \bigotimes_{k=1}^{n-1} \rho_k)) \otimes \rho_n)]$$

$$= \mathrm{Tr}_{\mathcal{H}_{A_n}} [e^{\tau L_n} (\rho_C^{(n-1)} \otimes \rho_n)].$$

Define the operator $\mathcal{L}$ as follows

$$\mathcal{L}(\rho) = \mathrm{Tr}_{\mathcal{H}_{A_n}}(e^{\tau L_n}(\rho \otimes \rho_n)).$$

Then the state of the cavity at the time $t = n\tau$ is the n-th power of $\mathcal{L}$

$$\rho_C(t) = \rho_C^{(n)} = \mathcal{L}(\rho_C^{(n-1)}) = \mathcal{L}^n(\rho_C). \tag{1}$$

And the state of the cavity at any time $t = n\tau + \nu \in [n\tau, (n+1)\tau]$ is

$$\rho_C(t) = \mathrm{Tr}_{\mathcal{H}_{A_{n+1}}}[e^{\nu L_{n+1}}(\mathcal{L}^n(\rho_C) \otimes \rho_{n+1})].$$

We will concentrate on the time of the form $t = n\tau$ for now.

Ideal cavity ($\sigma = 0$, no leakage)

Suppose that $t = n\tau$. Then the operator $\mathcal{L}$ is

$$\mathcal{L}(\rho) = \mathrm{Tr}_{\mathcal{H}_{A_n}}[e^{-i\tau H_n}(\rho \otimes \rho_n)e^{i\tau H_n}].$$

For the state ρ_n denote $p = \mathrm{Tr}(a_n^* a_n \rho_n)$.
Our first interest is the number of photons $N = b^* b$ in the cavity, which at the time t can be expressed as

$$N(t) = \mathrm{Tr}_{\mathcal{H}_C}(b^* b \rho_C(t)) = \mathrm{Tr}_{\mathcal{H}_C}(b^* b \mathcal{L}^n(\rho_C)).$$

Theorem

Let ρ_C be a gauge invariant state. The number of photons in the cavity at time $t = n\tau$ is

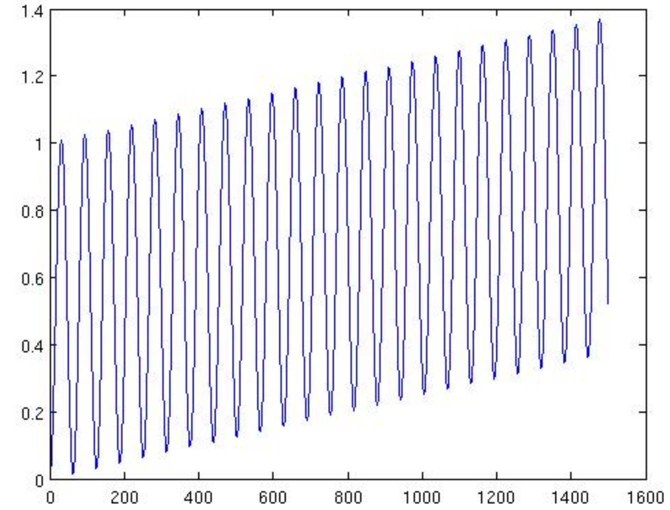
$$N(t) = N(0) + n\,p(1-p)\,\frac{2\lambda^2}{\epsilon^2}\,(1-\cos\epsilon\tau) + p^2\frac{2\lambda^2}{\epsilon^2}(1-\cos n\epsilon\tau)\;.$$

Remark

The theorem shows that only flux of randomly exited atoms ($0 < p < 1$) is able to produce a pumping of the cavity by photons.

Here $p = 1/2, \tau = 0.1, \lambda = \epsilon = 0.1$.

1.jpg



Leaking cavity ($\sigma > 0$)

Theorem

Suppose the initial state of the cavity is gauge invariant. Then the number of photons in the cavity in time converges to

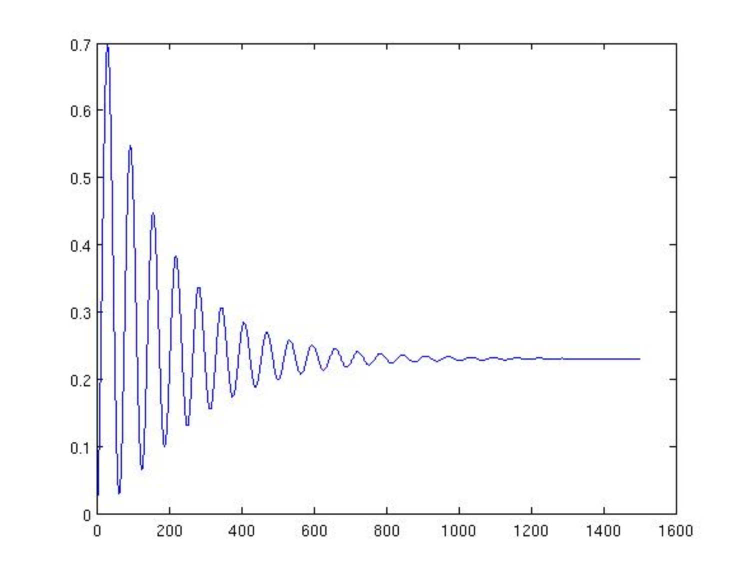
$$p\frac{\lambda^2}{|\mu|^2}\frac{1+e^{-\sigma\tau}-2e^{-\frac{\sigma}{2}\tau}\cos\epsilon\tau}{1-e^{-\sigma\tau}} - p^2\frac{2\lambda^2}{|\mu|^2}\frac{e^{-\sigma\tau}-e^{-\frac{\sigma}{2}\tau}\cos\epsilon\tau}{1-e^{-\sigma\tau}}.$$

Remark

If we consider $\sigma = 0$ we get that $N(t) = \frac{\lambda^2}{\epsilon^2}$. Which means that limits n goes to infinity and σ goes to zero do not commute.

Here $p = 1/2, \tau = 0.1, \lambda = \epsilon = 0.1, \sigma = 0.01$

3.jpg



Limiting state

A state is considered to be a linear functional

$$\omega_t^C(\cdot) = \mathrm{Tr}_{\mathcal{H}_C}(\cdot \rho_C(t)).$$

To study the limiting state we consider it on the space of the Weyl operators

$$\omega_C(W(\alpha)) = \lim_{n\to\infty} \omega_t^C(W(\alpha)).$$

Here for any complex number $\alpha \in \mathbb{C}$ the Weyl operator is defined as a unitary operator on the Hilbert space of the cavity $\mathcal{H}_C$ as follows

$$W(\alpha) = e^{\alpha b - \overline{\alpha} b^*}.$$

Theorem

The limiting state of the system exists in a weak- limit on the space of Weyl operators and it is independent of the initial state.*

When $t = n\tau$ we have

$$\begin{aligned}\omega_t^C(W(\alpha)) &= \mathrm{Tr}(W(\alpha)\rho_C(t)) = \mathrm{Tr}(W(\alpha)\mathcal{L}^n(\rho_C)) \\ &= \mathrm{Tr}((\mathcal{L}^*)^n(W(\alpha))\rho_C).\end{aligned}$$

We show that $(\mathcal{L}^*)^n(W(\alpha))$ has a limit when n goes to infinity, which guarantees the weak$-*$ limit of the states $\omega_t^C(\cdot)$ in the limit $t \to \infty$.

Theorem

The limiting state is a not a quasi-free state.

The quasi-free state ω satisfies

$$\omega(W(\alpha)) = \exp\{i\omega(b(\alpha)) - \frac{1}{2}(\omega(b(\alpha)^2) - \omega(b(\alpha))^2)\},$$

where $b(\alpha) = -i\alpha b + i\overline{\alpha}b^*$.
We show that the above equality does not hold.

Corollary

The limiting state is a non-equilibrium state.

Thank you!