Weak Localization in the alloy-type 3D Anderson Model

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Outline

- Background
- Statement of the result
- Proof of weak localization
- Wegner's estimate
- Proof of lemma

Setup

Consider the random Schrödinger operator of the following type

$$(H_\omega^\lambda \psi)(x) \; := \; -\frac{1}{2}(\Delta\psi)(x) + \lambda V_\omega(x)\psi(x) \, .$$

Here Δ denotes the discrete Laplace operator,

$$(\Delta\psi)(x) \; = \sum_{e \in \mathbb{Z}^3, \; |e|=1} \psi(x+e) \; - \; 6\psi(x) \, ,$$

and V_ω stands for a random multiplication operator of the form

$$V_\omega(x) \; = \; \sum_{i \in \mathbb{Z}^3} \omega_i \, u(x-i) \, .$$

Note the spectrum of the unperturbed operator H_w^0 is absolutely continuous and is the interval $[0, 6]$.

Background

- For $3D$ models, the first weak localization result for the Anderson model was proved by Frölich and Spencer(1983) through multiscale analysis.
- There are many results quantifying the weak localization region when the single cite potential is a delta function. For example, Aizenman(1994) shows there is localization in the region $[-a\lambda, -a\lambda + \lambda^{5/4}]$. Klopp(2002) derives a upper bound on the of order $-\lambda^{7/6}$. Elgart(2009) pushes the upper bound to $-\lambda^2$ by using a Feynman diagrammatic techinque.
- This work extends the results in Elgart(2009) to a general single cite potential. Related work on diagramatic techniques includes Erdos and Yau(2000) and Chen(2005). Notice non-monotonicity of the single cite potential poses a problem in deriving Wegner's estimate. Also there are issues involving employing the diagramatic technique.

Assumptions

- u decays exponentially fast:

$$|u(x)| \leq Ce^{-A|x|}$$

- u is compactly supported.
- The random variables $\{\omega_i\}$ are independent, identically distributed, even, and compactly supported on an interval J, with bounded probability density ρ. Moreover, function ρ is Lipschitz continuous:

$$|\rho(x) - \rho(y)| \leq K|x - y|$$

- The moments of ω_i satisfy

$$\mathbb{E}[\omega_i^{2m}] = \tilde{c}_{2m} \leq (2m)!c_v, \quad \tilde{c}_2 = 1, \quad \forall i \in \mathbb{Z}^3, m \in \mathbb{N}$$

Results

Let $\hat{u}$ denotes the Fourier transform of u.

$$\hat{u}(p) = \sum_{n \in \mathbb{Z}^3} e^{-i2\pi p \cdot n} u(n), \quad p \in \mathbb{T}^3 = [-1/2, 1/2]^3$$

Spectral localization

$$E_0 = -2\lambda^2 ||\hat{u}||_\infty^2 - 2\lambda^4 ||\hat{u}||_\infty^4 \tag{1}$$

For any $\alpha > 0$ there exists $\lambda_0(\alpha)$ such that for all $\lambda < \lambda_0(\alpha)$ the spectrum of H_ω within the set $E < E_0 - \lambda^{4-\alpha}$ is almost surely of the pure-point type, and the corresponding eigenfunctions are exponentially localized.

Lemma 1

For any integer N and energies E that satisfy the condition of above theorem we have the decomposition

$$R(x,y) \;=\; \sum_{n=0}^{N-1} A_n(x,y) \;+\; \sum_{z\in Z^3} \tilde{A}_N(x,z)R(z,y)\,,$$

with $A_0(x,y) = R_r(x,y)$, and where the (real valued) kernels A_n, $\tilde{A}_N$ satisfy bounds

$$\mathbb{E}|A_n(x,y)|^2 \;\leq\; (4n)!\,E^* \left(C(E^*)\,\frac{\lambda^2}{\sqrt{E^*}} \right)^n e^{-\delta\,|x-y|}\,, \quad n \geq 1\,;$$

$$\mathbb{E}|\tilde{A}_N(x,y)| \;\leq\; \sqrt{(4N)!} \left(C(E^*)\,\frac{\lambda^2}{\sqrt{E^*}} \right)^{N/2} e^{-\delta\,|x-y|/2}\,, \quad N > 1\,;$$

where $\delta := \sqrt{E_0 - E - E^*}/(\sqrt{6}\pi)$.
The zero order contribution A_0 satisfies

$$|A_0(x,y)| \;\leq\; 2\,e^{-\frac{\delta}{3\sqrt{3}}|x-y|} \tag{2}$$

for all $x, y \in \mathbb{Z}^3$.

Proof of the theorem

Using above lemma, choosing $(4N)^4 = \frac{\sqrt{E^*}}{C(E^*)\,\lambda^2}$, we get a bound

$$\mathbb{E}|R(E+i\epsilon;x,k)| \;\le\; C\left(e^{-\delta\,L/2} + \frac{e^{-N}}{\epsilon\,\delta}\right).$$

Hence we obtain

$$\begin{aligned}
&\mathbb{E}|R_{\Lambda_{L,x}}(E+i\epsilon;x,w)| \\
&\le\; \mathbb{E}|R(E+i\epsilon;x,w)| \,+\, \mathbb{E}|R_{\Lambda_{L,x}}(E+i\epsilon;x,w) \,-\, R(E+i\epsilon;x,w)| \\
&\le\; C\,\frac{L^2}{\epsilon}\max_{\mathrm{dist}(k,\partial\Lambda)\le 1}\mathbb{E}|R(E+i\epsilon;x,k)| \\
&\le\; C\,\frac{L^2}{\epsilon}\left[e^{-\delta\,L/2} + \frac{e^{-N}}{\epsilon\delta}\right]. \qquad (3)
\end{aligned}$$

Let $I = [E-\epsilon^{1/4}, E+\epsilon^{1/4}]$, and consider two events, the first one is $G_\omega(I) := \left\{\omega \in \Omega :\ \sigma(H_{\Lambda_{L,x}}) \cap I = \emptyset\right\}$, the other one is $\sigma(H_{\Lambda_{L,x}}) \cap I \ne \emptyset$ For the first part, since

$$\left|R_{\Lambda_{L,x}}(E+i\epsilon;x,w) - R_{\Lambda_{L,x}}(E;x,w)\right| \le \epsilon^{1/2}.$$

Proof of the theorem

Pairing this bound with (3) and using Chebyshev's inequality

$$\mathrm{Prob}\left\{\omega \in G_\omega(I) : |R_{\Lambda_{L,x}}(E;x,w)| \geq C\frac{L^2}{\epsilon^{5/4}}\left[e^{-\delta L/2} + \frac{e^{-N}}{\epsilon\delta}\right] + \epsilon^{1/4}\right\} \leq C\epsilon^{1/4}. \quad (4)$$

The Wegner estimate implies that for the second event

$$\mathrm{Prob}\{\sigma(H_{\Lambda_{L,x}}) \cap I \neq \emptyset\} \leq C\,|I|\,|\Lambda_{L,x}|^2\,D^{-3/2} = C\,\epsilon^{1/4}\,D^{-3/2}\,L^4. \quad (5)$$

Combining (4) and (5) we arrive at

$$\mathrm{Prob}\left\{|R_{\Lambda_{L,x}}(E;x,w)| \geq C\frac{L^2}{\epsilon^{4/3}}\left[e^{-\delta L/2} + \frac{e^{-N}}{\epsilon\delta}\right] + \epsilon^{1/4}\right\} \leq C\epsilon^{1/4}D^{-3/2}L^4.$$

Choose $E^* = \lambda^{4-\alpha}/2$, $L = \lambda^{-2}$ and $\epsilon = e^{-4\lambda^{-B'\alpha/2}}$ we get the following initial volume estimate

$$\mathrm{Prob}\left\{|R_{\Lambda_{\lambda^{-2},x}}(E;x,w)| \geq e^{-\lambda^{-B'\alpha/2}}\right\} \leq e^{-\lambda^{-B'\alpha/2}}. \quad (6)$$

Wegner's estimate

Wegner's estimate

Let I be an open interval of energies such that

$$D \;:=\; \mathrm{dist}(I, \sigma(H^0_\omega)) \;>\; 0\,.$$

Then we have

$$\mathbb{E}\,\mathrm{tr} P_I(H^{\Lambda,\lambda}_\omega) \;\leq\; C\,|I|\,|\Lambda|^2\,D^{-3/2}\,, \tag{7}$$

where $H^{\Lambda,\lambda}_\omega$ denotes a natural restriction of H^λ_ω to $\Lambda \subset \mathbb{Z}^3$.

This Wegner's estimate may not be the optimal one, as in the δ potential case, where it is Λ instead of Λ^2. But suffice for the proof of localization.

Proof of Wegner's estimate

Notice

$$\mathbb{E}F_\omega = \int F_\omega \prod_{i\in\Lambda} \rho(\omega_i)d\omega_i = \frac{1}{2\delta}\int_{1-\delta}^{1+\delta} v^{|\Lambda|}dv \int F_{v\hat{\omega}} \prod_{i\in\Lambda} \rho(v\hat{\omega}_i)d\hat{\omega}_i\,.$$

$$\mathbb{E}\,\mathrm{tr}\,Im\left(H_\omega^{\Lambda,\lambda} - E - i\eta\right)^{-1}$$

$$\leq \frac{e}{2\delta}\int_{1-\delta}^{1+\delta} dv \int \frac{\eta}{v^2}\,\mathrm{tr}\,\left((v^{-1}A + V_{\hat{\omega}})^2 + (\eta/2)^2\right)^{-1} \prod_{i\in\Lambda} \rho(v\hat{\omega}_i)d\hat{\omega}_i\,.$$

where $A = \Delta_\Lambda - E$ is positive. Change variable $u = v^-1$, and factor out $A = A^{1/2}\cdot A^{1/2}$. We get

$$Tr\left((uA + V_{\hat{\omega}})^2 + (\eta/2)^2\right)^{-1}$$

$$\leq \|A^{-1}\|\;Tr\left(E^*(u + A^{-1/2}V_{\hat{\omega}}A^{-1/2})^2 + \frac{\eta^2}{4\|A\|}\right)^{-1}. \quad (8)$$

Integrate over u first and then $d\hat{w}_i$'s. We finally get

$$\mathbb{E}\,\mathrm{tr}P_I(H_\omega^{\Lambda,\lambda}) \leq C\,|I|\,|\Lambda|^2\,(E^*)^{-3/2}.$$

Outline of the proof of Lemma 1

- Resolvent expansion near the self-energy, and the range of localization follows from the bound on the self-energy.
- Renormalization (cancellation of tadpoles).
- Extraction of exponential decay.
- Diagramatic estimation on the resulting integral.

Resolvent expansion

Decompose H_ω^λ as

$$H_\omega^\lambda = H_r + \tilde{V}\,, \quad H_r := -\frac{1}{2}\Delta - \sigma(p, E+i\epsilon)\,, \quad \tilde{V} := \lambda V_\omega + \sigma(p, E+i\epsilon)\,,$$

Let $R_r := (H_r - E - i\epsilon)^{-1}$ We can expand R into resolvent series

$$R = \sum_{i=0}^{n} (-R_r\tilde{V})^i R_r \ + \ (-R_r\tilde{V})^{n+1} R\,. \tag{9}$$

Stop expansion term by term at order N. An order of a term is the number of appearances of σ and V_ω, where σ counts as 2. Thus $R_r\sigma R_r\lambda V_\omega R_r\sigma R$ has order 5. The following is the expansion for $N = 2$ following this stopping rule.

$$\begin{aligned} R \ = \ R_r \ - \ R_r\sigma R \ - \ \{\lambda R_r V_\omega R\} \ = \\ R_r \ - \ R_r\sigma R \ - \ \lambda R_r V_\omega R_r \\ + \ \lambda R_r V_\omega R_r \sigma R \ + \ \lambda^2 R_r V_\omega R_r V_\omega R\,, \end{aligned}$$

An example

The advantage of the above stopping rule is that all tadpoles will be cancelled. The following example illustrates this idea. Consider all the terms of order 4 in the expansion.

$$\lambda^4 R_r V_\omega R_r V_\omega R_r V_\omega R_r V_\omega R_r, \quad -\lambda^2 R_r V_\omega R_r V_\omega R_r \sigma R_r,$$
$$-\lambda^2 R_r V_\omega R_r \sigma R_r V_\omega R_r, \quad -\lambda^2 R_r \sigma R_r V_\omega R_r V_\omega R_r, \quad R_r \sigma R_r \sigma R_r$$

The expectation of the product of random variables will give us some delta functions.

$$\begin{aligned}
&\mathbb{E}[\omega(x_1)\omega(x_2)\omega(x_3)\omega(x_4)] \\
&= (1-\delta(x_1-x_3))\delta(x_1-x_2)\delta(x_3-x_4) + (1-\delta(x_1-x_2))\delta(x_1-x_3)\delta(x_2-x_4) \\
&\quad + (1-\delta(x_1-x_3))\delta(x_1-x_4)\delta(x_2-x_3) + \tilde{c}_4\delta(x_1-x_2)\delta(x_3-x_4)\delta(x_1-x_3) \\
&= \delta(x_1-x_2)\delta(x_3-x_4) + \delta(x_1-x_3)\delta(x_2-x_4) + \delta(x_1-x_4)\delta(x_2-x_3) \\
&\quad + (\tilde{c}_4-3)\delta(x_1-x_2)\delta(x_3-x_4)\delta(x_1-x_3)
\end{aligned}$$

Set $c_4 = \tilde{c}_4 - 3$, notice $\sigma = \lambda^2 R_r(x,x)$

An example

$$\mathbb{E}\langle \lambda^4 x R_r V_\omega R_r V_\omega R_r V_\omega R_r V_\omega R_r y\rangle$$
$$= \sum_{x_1,x_2,x_3,x_4} \langle x R_r x_1\rangle\langle x_1 R_r x_2\rangle\langle x_2 R_r x_3\rangle\langle x_3 R_r x_4\rangle\langle x_4 R_r y\rangle \mathbb{E}[\omega_{x_1}\omega_{x_2}\omega_{x_3}\omega_{x_4}]$$
$$= \sigma^2\langle x R_r^3 y\rangle + \lambda^4 \sum_{x_1,x_2} \langle x R_r x_1\rangle\langle x_1 R_r x_2\rangle\langle x_2 R_r x_1\rangle\langle x_1 R_r x_2\rangle\langle x_2 R_r y\rangle$$
$$+ \lambda^2\sigma \sum_{x_1} \langle x R_r x_1\rangle\langle x_1 R_r^2 x_1\rangle\langle x_1 R_r y\rangle + c_4\lambda^4 \sum_{x_1} \langle x R_r x_1\rangle\langle x_1 R_r x_1\rangle^3\langle x_1 R_r x_1\rangle$$

The tadpoles are the first term and third terms. The first term is equal to $\mathbb{E}\langle x\lambda^2 R_r V_\omega R_r V_\omega R_r \sigma R_r y\rangle$, $\mathbb{E}\langle x\lambda^2 R_r \sigma R_r V_\omega R_r V_\omega R_r y\rangle$, and $\mathbb{E}\langle x R_r \sigma R_r \sigma R_r y\rangle$. The third term is equal to $\mathbb{E}\langle x\lambda^2 R_r V_\omega R_r \sigma R_r V_\omega R_r y\rangle$. So they cancel out exactly. This is the case when the single cite function is the delta function.

An example

When the single cite potential is of the more general form

$$V_\omega(x) = \lambda \sum_{i \in \mathbb{Z}^d} q_i(\omega) u(x-i),$$

we represent it in its Fourier transform. Using

$$R_r(z,w) = \int e^{i2\pi(z-p)} \frac{d^3p}{E(p)}, \quad \hat{V}_\omega(p) = \hat{u}(p)\hat{\omega}(p),$$

We get for $\mathbb{E}\langle \lambda^4 x R_r V_\omega R_r V_\omega R_r V_\omega R_r V_\omega R_r y \rangle$

$$\int_{(\mathbb{T}^3)^5} e^{2\pi i(p_1 x - p_5 y)} \prod_{i=1}^{5} \frac{d^3 p_i}{E(p_i)} \prod_{i=1}^{4} \hat{u}(p_i - p_{i+1})$$
$$\mathbb{E}\left[\hat{\omega}(p_1 - p_2)\hat{\omega}(p_2 - p_3)\hat{\omega}(p_3 - p_4)\hat{\omega}(p_4 - p_5)\right]$$

and the renormalization process goes through.

General case

Some combinatorics will show the renormalization is true for general N. Without concerning too much details of the notation, we present the following identity.

$$\mathbb{E}\left[\prod_{j\in\Upsilon_{N,N}} \omega_{x_j}\right] = \sum_{m=1}^{N} \sum_{\pi=\{S_j\}_{j=1}^m} \prod_{j=1}^{m} c_{|S_j|}\delta(x_{S_j}),$$

where

$$\delta(x_S) = \sum_{y\in\mathbb{Z}^3} \prod_{j\in S} \delta_{|x_j-y|},$$

and $c_{2l} \leq (cl)^{2l+1}$, $c_2 = \mathbb{E}\,\omega_x^2 = 1$. If S_j in $\pi \in \Pi$ has form $S_j = \{i, i+1\}$, we refer to it as a tadpole. Then the following lemma is a result of the following identity

$$\sum_{k=0}^{N}(-1)^k \sum_{\substack{\pi\in\Pi:\\ \pi=\pi_k^c\cup\{S\}}} \mathbb{E}\left[\prod_{i\in S}\omega_{x_i}\right] \prod_{S_l\in\pi_k^c} \delta(x_{S_l}) = \sum_{\substack{\pi\in\Pi:\\ \pi=\pi_0}} \prod_{S_j\in\pi} c_{|S_j|}\delta(x_{S_j}).$$

Lemma 2

For A_l defined in Lemma 1, the function $\mathbb{E}\,|A_l(x,y)|^2$ is a function of the variable $x-y$. Let

$$\mathcal{A}_{l,E}(x-y) \;:=\; \mathbb{E}\,|A_l(x,y)|^2\,,$$

then we have

$$\mathbb{E}|\mathcal{A}_l(x,y)|^2 = \lambda^{2l}\int_{(\mathbb{T}^3)^{2l+2}} e^{i\alpha}\,\frac{dp_{l+1}}{E(p_{l+1})}\,\frac{dp_{2l+2}}{E(p_{2l+2})}\prod_{j=1}^{l}\frac{dp_j}{E(p_j)}\prod_{j=l+2}^{2l+1}\frac{dp_j}{E^*(p_j)}$$

$$\times\prod_{i\in\Upsilon_l}\hat{u}(p_j-p_{j+1})\sum_{\substack{\pi\in\Pi_l:\\ \pi=\pi_0}}\prod_{S_k\in\pi}c_{|S_k|}\,\delta\left(\sum_{i\in S_k}p_i-p_{i+1}\right)\,, \quad (10)$$

Proof of the Lemma 1

We first want to establish the exponential decay of $\mathcal{A}_{l,E}(x-y)$ in $|x-y|$ We will show that for a general value of l,

$$\mathcal{A}_{l,E}(x) \ \leq \ \|\hat{u}\|_{\infty,\mathcal{R}}^{2l} \cdot e^{-\sqrt{\delta/3}\,|x|} \ \hat{\mathcal{A}}_{l,E^*}(0)\,, \qquad (11)$$

where

$$\hat{\mathcal{A}}_{l,E^*}(0) := \lambda^{2l} \int_{(\mathbb{T}^3)^{2l+2}} e^{i\alpha} \, \frac{dp_{l+1}}{e(p_{l+1})+E^*} \, \frac{dp_{2l+2}}{e(p_{2l+2})+E^*} \prod_{j\in\Upsilon_l} \frac{dp_j}{e(p_j)+E^*} \times \sum_{\substack{\pi\in\Pi_l: \\ \pi=\pi_0}} \prod_{S_k\in\pi} c_{|S_k|} \, \delta\left(\sum_{i\in S_k} p_i - p_{i+1}\right) . \qquad (12)$$

The expression $\hat{\mathcal{A}}_{l,E^*}(0)$ has been studied in Elgart(2009) before and shown that

$$\hat{\mathcal{A}}_{l,E^*}(0) \ \leq \ (4l)!\, E^* \left(C \ln^9(E^*) \, \frac{\lambda^2}{\sqrt{E^*}} \right)^l . \qquad (13)$$

Proof of the Lemma 1

Using the delta function introduced before and integrate over the tree momenta. Let $E(p) = e(p) - E - i\epsilon - \sigma(p, E + i\epsilon)$.

$$
\begin{aligned}
\mathcal{A}_{l,E}(x) \; &= \; \lambda^{2l} \sum_{\pi} c_{\pi} \int dw_1 \, e^{-i2\pi w_1 \cdot x} \prod_{i=1}^{r_\pi} \frac{1}{E^{\sharp}(w_1 + q_i)} \prod_{j=1}^{s_\pi} \hat{u}(w_1 + Q_j) \\
&\times \int e^{-i2\pi w_2 \cdot x} \prod_{t \in \Phi'} dw_t \prod_{i=r_\pi+1}^{2n+2} \frac{1}{E^{\sharp}(q_i)} \prod_{j=s_\pi+1}^{2n} \hat{u}(Q_j) \, ,
\end{aligned}
$$

where $E^{\sharp}(p)$ stands for either $E(p)$ or $E^*(p)$, Φ' is a set of indices of loop momentum that does not include w_1, and q_i, Q_j are some linear combinations of the loop variables in Φ'. Note now that the integral with respect to w_1 becomes

$$
\begin{aligned}
&\int dw_1^{\perp} \, e^{-i2\pi (w_1 \cdot x - (w_1 \cdot e_\gamma) x_\gamma)} \times \\
&\int_{-1/2}^{1/2} d(w_1 \cdot e_\gamma) \prod_{i=1}^{r_\pi} \frac{1}{E^{\sharp}(w_1 + q_i)} e^{-i2\pi (w_1 \cdot e_\gamma) x_\gamma} \prod_{j=1}^{s_\pi} \hat{u}(w_1 + Q_j) \qquad (14)
\end{aligned}
$$

Proof of the Lemma 1

The exponential decay is established by extending the second part of (14) to complex coordinate. Let $\mathcal{R}$ denote

$$\{ -1/2 \pm i\sqrt{\delta};\ 1/2 \pm i\sqrt{\delta}\} .$$

This integral is periodic over vertical segments of $\mathcal{R}$ and therefore

$$\left| \int_{\mathbb{T}} d(w_1 \cdot e_\gamma) \prod_{i=1}^{r_\pi} \frac{1}{E^\sharp(w_1 + q_i)}\, e^{-i2\pi x_\gamma\,(w_1 \cdot e_\gamma)} \prod_{j=1}^{s_\pi} \hat{u}(w_1 + Q_j) \right|$$
$$= \left| \int_{\mathbb{T} - i\sqrt{\delta}} d(w_1 \cdot e_\gamma) \prod_{i=1}^{r_\pi} \frac{1}{E^\sharp(w_1 + q_i)}\, e^{-i2\pi x_\gamma\,(w_1 \cdot e_\gamma)} \prod_{j=1}^{s_\pi} \hat{u}(w_1 + Q_j) \right|$$
$$\leq \ \|\hat{u}\|_{\infty, E^*}^{s_\pi} \cdot e^{-|x|\sqrt{E^*/3}} \int_{\mathbb{T}} d(w_1 \cdot e_\gamma) \prod_{i=1}^{r_\pi} \frac{1}{e(w_1 + q_i) + E^*}\,, \tag{15}$$

Now we have arrived at (11).

Properties of self-energy $\sigma(p, E)$

Recall the self energy term σ, associated with H_ω^λ, is given by the solution of the self-consistent equation

$$\sigma(p, E + i\epsilon) \;=\; \lambda^2 \int_{\mathbb{T}^3} d^3q \, \frac{|\hat{u}(p-q)|^2}{e(q) - E - i\epsilon - \sigma(q, E + i\epsilon)} \, . \qquad (16)$$

We need existence, periodicity, and analyticity of the self energy operator $\sigma(p, E + i\epsilon)$. Consider space

$$L(\mathbb{T}^3) \;=\; \{f \,:\, \mathbb{T}^3 \to \mathbb{C} \,\big|\, \|f\|_\infty < \infty \,, f \;\; \text{is real analytic}\} \, .$$

and define map $T_\epsilon \,:\, L(\mathbb{T}^3) \to L(\mathbb{T}^3)$ pointwise as

$$(T_\epsilon f)(p) = \lambda^2 \int_{\mathbb{T}^3} d^3q \, \frac{|\hat{u}(p-q)|^2}{e(q) - E - i\epsilon - f(q)} \, . \qquad (17)$$

Then T_ϵ is a contraction on the ball $B_\beta(0)$ where $\beta = 2\lambda^2 ||\hat{u}||_\infty^2$ for all p, ϵ and $E \le E_0 = -2\lambda^2 \|\hat{u}\|_\infty^2 - 2\lambda^4 \|\hat{u}\|_\infty^4$.

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Thank you

Thank you!