The following problems are designed to help you study for the first in-class exam. Problems may or may not be an accurate indicator of the types of problems you can expect to see on the exam.

1. Find the unit vector in the direction of $0.06\hat{i} - 0.08\hat{k}$

2. Find the unit vector in the opposite direction to $2\hat{i} - \hat{j} - \sqrt{11}\hat{k}$.

3. Find a vector of length 7 that points in the same direction as $\hat{i} - \hat{j} + 2\hat{k}$.

4. Resolve the vectors below into components:

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6. From the figure below, read off the coordinates of the five points, $A$, $B$, $C$, $D$, $E$, and thus resolve into components the following two vectors: $\vec{u} = (2.5)\vec{AB} + (-0.8)\vec{CD}$, and $\vec{v} = (2.5)\vec{BA} - (-0.8)\vec{CD}$. What is the relationship between $\vec{u}$ and $\vec{v}$? Why was this to be expected?

7. The figure below shows a molecule with four atoms at $O$, $A$, $B$ and $C$. Check that every atom in the molecule is 2 units away from every other atom.
8. A large ship is being towed by two tugs. The larger tug exerts a force which is 25% greater than the smaller tug and at an angle of 30 degrees north of east. Which direction must the smaller tug pull to ensure that the ship travels due east?

9. An object \( P \) is pulled by a force \( \vec{F}_1 \) of magnitude 15 lb at an angle of 20 degrees north of east. In what direction must a force \( \vec{F}_2 \) of magnitude 20 lb pull to ensure that \( P \) moves due east?

10. An airplane heads northeast at an airspeed of 700 km/hr, but there is a wind blowing from the west at 60 km/hr. In what direction does the plane end up flying? What is its speed relative to the ground?

11. Consider the following vectors:
\[ \vec{a} = 2\hat{j} + \hat{k}, \quad \vec{b} = -3\hat{i} + 5\hat{j} + 4\hat{k}, \quad \vec{c} = \hat{i} + 6\hat{j}, \]
\[ \vec{y} = 4\hat{i} - 7\hat{j}, \quad \vec{z} = \hat{i} - 3\hat{j} - \hat{k}. \]
Perform the following operations:

(a) \( \vec{a} \cdot \vec{y} \)
(b) \( \vec{c} \cdot \vec{y} \)
(c) \( \vec{a} \cdot \vec{b} \)
(d) \( \vec{a} \cdot \vec{z} \)
(e) \( \vec{c} \cdot \vec{a} + \vec{a} \cdot \vec{y} \)
(f) \( \vec{a} \cdot (\vec{c} + \vec{y}) \)
(g) \( (\vec{a} \cdot \vec{b})\vec{a} \)
(h) \( (\vec{a} \cdot \vec{y})(\vec{c} \cdot \vec{z}) \)
(i) \( ((\vec{c} \cdot \vec{c})\vec{a}) \cdot \vec{a} \)
12. A 100-meter dash is run on a track in the direction of the vector \( \vec{v} = 2\hat{i} + 6\hat{j} \). The wind velocity \( \vec{w} \) is \( 5\hat{i} + \hat{j} \) km/hr. The rules say that a legal wind speed measured in the direction of the dash must not exceed 5 km/hr. Will the race results be disqualified due to an illegal wind? Justify your answer.

13. An airplane is flying toward the southeast. Which of the following wind velocity vectors increases the plane’s speed the most? Which slows down the plane the most?

\[
\vec{w}_1 = -4\hat{i} - \hat{j}, \quad \vec{w}_2 = \hat{i} - 2\hat{j}, \quad \vec{w}_3 = -\hat{i} + 8\hat{j}, \\
\vec{w}_4 = 10\hat{i} + 2\hat{j}, \quad \vec{w}_5 = 5\hat{i} - 2\hat{j}.
\]

14. A canoe is moving with velocity \( \vec{v} = 5\hat{i} + 3\hat{j} \) m/sec relative to the water. The velocity of the current in the water is \( \vec{c} = \hat{i} + 2\hat{j} \) m/sec.

(a) What is the speed of the current?
(b) What is the speed of the current in the direction of the canoe’s motion?

15. A force of 15 N, pointing in the direction of 30° north of east acts on a particle which displaces over a distance of 5 m in a direction of 60° north of east. Find the work done on the particle by the force.

16. Show that the vectors \((\vec{b} \cdot \vec{c})\vec{a} - (\vec{a} \cdot \vec{c})\vec{b} \) and \( \vec{c} \) are perpendicular.

17. For any two vectors \( \vec{v} \) and \( \vec{w} \), consider the following function of \( t \):

\[
q(t) = (\vec{v} + t\vec{w}) \cdot (\vec{w} + t\vec{w})
\]

(a) Explain why \( q(t) \geq 0 \) for all real \( t \).
(b) Expand \( q(t) \) as a quadratic polynomial in \( t \) using the properties of the dot product.
(c) Using the discriminant of the quadratic, show that

\[
|\vec{v} \cdot \vec{w}| \leq ||\vec{v}|| \cdot ||\vec{w}||.
\]
18. Find $\vec{v} \times \vec{w}$ for the following cases:

(a) $\vec{v} = \hat{k}$, $\vec{w} = \hat{j}$.
(b) $\vec{v} = -\hat{i}$, $\vec{w} = \hat{j} + \hat{k}$.
(c) $\vec{v} = \hat{i} + \hat{k}$, $\vec{w} = \hat{i} + \hat{j}$.
(d) $\vec{v} = \hat{i} + \hat{j} + \hat{k}$, $\vec{w} = \hat{i} + \hat{j} - \hat{k}$.
(e) $\vec{v} = 2\hat{i} - 3\hat{j} + \hat{k}$, $\vec{w} = \hat{i} + 2\hat{j} - \hat{k}$.
(f) $\vec{v} = -3\hat{i} + 5\hat{j} + 4\hat{k}$, $\vec{w} = \hat{i} - 3\hat{j} - \hat{k}$.

19. Use the properties of the cross product to find:

(a) $((\hat{i} + \hat{j}) \times \hat{i}) \times \hat{j}$.
(b) $(\hat{i} + \hat{j}) \times (\hat{i} \times \hat{j})$.

20. Suppose $\vec{v} \cdot \vec{w} = 5$ and $||\vec{v} \times \vec{w}|| = 3$, and the angle between $\vec{v}$ and $\vec{w}$ is $\theta$. Find $\tan \theta$ and $\theta$.

21. Find the equation of a sphere with radius 2 and centered at $(1, 0, 0)$.

22. Find the equation of the vertical plane perpendicular to the $y$-axis and through the point $(2, 3, 4)$.

26. Consider the acceleration due to gravity, $g$, at a distance $h$ from the center of a planet of mass $m$.

(a) If $m$ is held constant, is $g$ an increasing or decreasing function of $h$? Why?

(b) If $h$ is held constant, is $g$ an increasing or decreasing function of $m$? Why?

27. Find a formula for the set of points whose distance from the line given by $z = 4$ in the $yz$-plane is equal to 7. [HINT: Draw a picture].

28. Find the equations of planes that just touch the sphere $(x-2)^2 + (y-3)^2 + (z-3)^2 = 16$ and are parallel to the coordinate planes (i.e. the $xy$-plane, the $yz$-plane, and the $xz$-plane).

29. Find a formula for the set of points whose distance from the $z$-axis equals the distance from the $xy$-plane.

30. Sketch a graph of the following functions, and briefly describe them in words:

(a) $x^2 + y^2 + z^2 = 9$
(b) $z = x^2 + y^2 + 4$
(c) $z = 5 - x^2 - y^2$
(d) $z = y^2$
(e) $2x + 4y + 3z = 12$
(f) $x^2 + z^2 = 4$
31. Consider the function \( f \) given by \( f(x, y) = y^3 + xy \). Draw graphs of cross-sections with:
   (a) \( x \) fixed at \( x = -1, x = 0, \) and \( x = 1 \).
   (b) \( y \) fixed at \( y = -1, y = 0, \) and \( y = 1 \).

32. By setting one variable constant, find a plane that intersects the graph of \( z = 4x^2 - y^2 + 1 \) in:
   (a) Parabola opening upward
   (b) Parabola opening downward
   (c) Pair of intersecting straight lines

33. For the following functions, sketch a contour diagram for the function with at least four labeled contours. Describe in words the contours and how they are spaced.
   (a) \( f(x, y) = x + y \)
   (b) \( f(x, y) = 3x + 3y \)
   (c) \( f(x, y) = x^2 + y^2 \)
   (d) \( f(x, y) = -x^2 - y^2 + 1 \)
   (e) \( f(x, y) = xy \)
   (f) \( f(x, y) = y - x^2 \)
   (g) \( f(x, y) = x^2 + 2y^2 \)
   (h) \( f(x, y) = \sqrt{x^2 + 2y^2} \)
   (i) \( f(x, y) = \cos \sqrt{x^2 + y^2} \)

34. Total sales, \( Q \), of a product are a function of its price and the amount spent on advertising. The figure below shows a contour diagram for total sales. Which axis corresponds to the price of the product and which to the amount spent on advertising? Explain.
35. The figure below shows contour diagrams of \( f(x,y) \) and \( g(x,y) \). Sketch the smooth curve with equation \( f(x,y) = g(x,y) \).

![Contour Diagrams](image)

36. Let \( f(x,y) = x^2 + y^2 \). Find the value of \( c \) if the contour \( f = c \) is to pass through the point \((\sqrt{2}/2,\sqrt{2}/2)\).

37. Find the equation of the plane which passes through \((1,1,1)\) and is perpendicular to the line of intersection of the planes \( x + y - z = 10 \) and \( 2x - y + z = 1 \).

38. Let \( A = (-1,3,0) \), \( B = (3,2,4) \), and \( C = (1,-1,5) \). Use the notion of the cross product to find an equation for the plane that passes through these three points.

39. A plane has equation \( z = 5x - 2y + 7 \). Find a value of \( \lambda \) making the vector \( \lambda \hat{i} + \hat{j} + 0.5\hat{k} \) normal to the plane.

40. Find a vector perpendicular to the plane \( z = 2x + 3y \).

41. Find the linear equation whose graph contains the points \((0,0,0)\), \((0,2,-1)\) and \((-3,0,4)\).

42. Use the catalog of surfaces to identify the following surfaces:
   
   (a) \( x^2 + y^2 - z = 0 \).
   
   (b) \(-x^2 - y^2 + z^2 = 1 \)
   
   (c) \( x + y = 1 \)
   
   (d) \( x^2 + y^2/4 + z^2 = 1 \).
43. Decide whether the following level surfaces can be expressed as the graph of a function \( f(x, y) \):

   (a) \( z - x^2 - 3y^2 = 0 \)
   (b) \( 2x + 3y - 5z - 10 = 0 \)
   (c) \( x^2 + y^2 + z^2 - 1 = 0 \)
   (d) \( z^2 = x^2 + 3y^2 \).

44. The surface \( S \) is the graph of \( f(x, y) = \sqrt{1 - x^2 - y^2} \). Explain why \( S \) is the upper hemisphere of radius 1, with equator in the \( xy \)-plane, centered at the origin. Find a level surface \( g(x, y, z) = c \) representing \( S \).

45. Use difference quotients with \( \Delta x = 0.1 \) and \( \Delta y = 0.1 \) to estimate \( f_x(1, 3) \) and \( f_y(1, 3) \) where

   \[ f(x, y) = e^{-x} \sin y. \]

   Then, give better estimates by using \( \Delta x = 0.01 \) and \( \Delta y = 0.01 \).

46. Approximate \( f_x(3, 5) \) using the contour diagram of \( f(x, y) \) shown below.

47. The figure below shows the saddle-shaped surface \( z = f(x, y) \). Determine the signs of \( f_x(0, 5) \) and \( f_y(0, 5) \).
48. Your monthly car payment in dollars is \( P = f(P_0, t, r) \), where \( P_0 \) is the amount you borrowed, \( t \) is the number of months it takes to pay off the loan, and \( r \% \) is the interest rate. What are the units, the financial meaning, and the signs of \( \partial P/\partial t \) and \( \partial P/\partial r \)?

49. Find the partial derivatives of the following functions.

(a) \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \) if \( z = (x^2 + x - y)^7 \)

(b) \( f_x \) and \( f_y \) if \( f(x, y) = A^\alpha x^{\alpha + \beta} y^{1-\alpha-\beta} \)

(c) \( \frac{\partial}{\partial y} (3x^5y^7 - 32x^4y^3 + 5xy) \)

(d) \( f_x(1, 2) \) and \( f_y(1, 2) \) if \( f(x, y) = x^3 + 3x^2y - 2y^2 \)

(e) \( \frac{\partial}{\partial t} e^{\sin(x+ct)} \)

(f) \( \frac{\partial}{\partial x} (a\sqrt{x}) \)

(g) \( g_x \) if \( g(x, y) = \ln(ye^{xy}) \)

(h) \( u_E \) if \( u = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0} B^2 \).

(i) \( \frac{\partial}{\partial x} \left( \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)} \right) \)

(j) \( \frac{\partial}{\partial \phi} (\sin(\pi\theta\phi) + \ln(\theta^2 + \phi)) \).

(k) \( \frac{\partial V}{\partial r} \) and \( \frac{\partial V}{\partial h} \) if \( V = \frac{4}{3}\pi r^2 h \).

(l) \( \frac{\partial Q}{\partial K} \) if \( Q = c(a_1K^{b_1} + a_2L^{b_2})^\gamma \).

(m) \( \frac{\partial f}{\partial x} \bigg|_{(\alpha/3, 1)} \) if \( f(x, y) = x \ln(y \cos x) \).

50. The surface \( S \) is given, for some constant \( a \), by

\[
z = 3x^2 + 4y^2 - axy.
\]

Find the values of \( a \) which ensure that \( S \) is sloping upward when we move in the positive \( x \)-direction from the point \((1, 2)\). With this value of \( a \), if you move in the positive \( y \)-direction from the point \((1, 2)\), does the surface slope up or down? Explain.
51. Find the equation of the tangent plane at the given point.
   (a) \( z = ye^{x/y} \) at the point \((1, 1, e)\)
   (b) \( z = \sin(xy) \) at the point where \( x = 2, \ y = 3\pi/4 \).
   (c) \( z = \ln(x^2 + 1) + y^2 \) at the point \((0, 3, 9)\)
   (d) \( z = e^y + x + x^2 + 6 \) at the point \((1, 0, 9)\).
   (e) \( z = \frac{1}{2}(x^2 + 4y^2) \) at the point \((2, 1, 4)\)
   (f) \( x^2 + y^2 - z = 1 \) at the point \((1, 3, 9)\).
   (g) \( x^2y + \ln(xy) + z = 6 \) at the point \((4, 0.25, 2)\).

52. For the following exercises, find the differential of the function.
   (a) \( f(x, y) = \sin(xy) \)
   (b) \( g(u, v) = u^2 + uv \)
   (c) \( z = e^{-x} \cos y \)
   (d) \( h(x, t) = e^{-3t} \sin(x + 5t) \).

53. A student was asked to find the equation of the tangent plane to the surface \( z = x^3 - y^2 \) at the point \((x, y) = (2, 3)\). The student’s answer was
   \[ z = 3x^2(x - 2) - 2y(y - 3) - 1. \]
   At a glance, how do you know this is wrong? What mistake did the student make? Answer the question correctly.