Section 13.1: Displacement Vectors

A displacement vector is a geometric object which encodes both a displacement and a direction.

The **displacement vector** from one point to another is an arrow with its tail at the first point and its tip at the second. The **magnitude** (or length) of the displacement vector is the distance between the points and is represented by the length of the arrow. The **direction** of the displacement vector is the direction of the arrow.

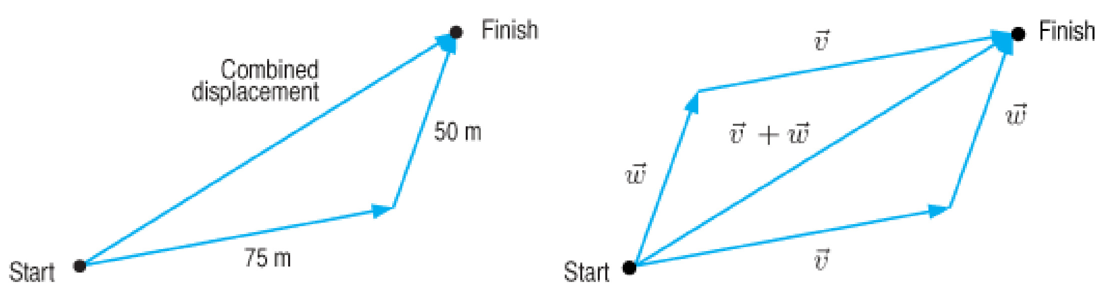
Note: Displacement vectors which point in the same direction and have the same magnitude are considered to be the same, even if they do not coincide.

Notation and Terminology: There are mainly two types of quantities we concern ourselves with: vectors and scalars. As you have seen, a *vector* has both magnitude and direction. A *scalar*, on the other hand, is a quantity specified only by a number, and has no direction.

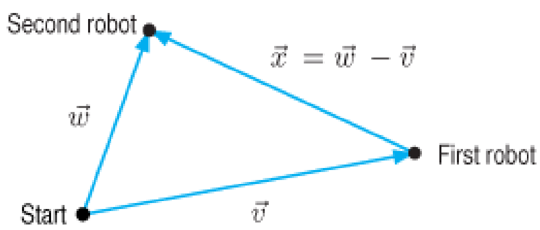
Throughout the course, we will denote vectors with an arrow. For example, $\vec{v}$ is a vector. A scalar will be denoted by simple italics. At times, we will use the notation $\vec{PQ}$ to denote the displacement vector from point P to point Q.

Addition and Subtraction of Vectors: The addition of two vectors can be easily conceptualized. Suppose that you start at a specific point and then move 25 feet to the east, then 25 feet north. Your displacement is then $25\sqrt{2}$ feet in a direction of $45°$ with respect to the horizontal.

The **sum**, $\vec{v}+\vec{w}$, of two vectors $\vec{v}$ $\vec{w}$ is the combined displacement resulting from first applying $\vec{v}$ and then $\vec{w}$. The sum $\vec{w}+\vec{v}$ gives the same displacement.



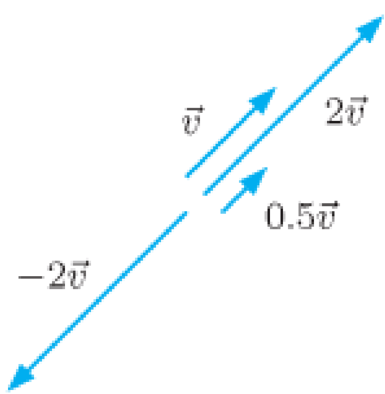
The **difference**, $\vec{w} - \vec{v}$, is the displacement vector that, when added to $\vec{v}$, gives $\vec{w}$. That is, $\vec{w} = \vec{v} + (\vec{w} - \vec{v})$.



The **zero vector**, $\vec{0}$, is a displacement vector with zero length.

Scalar Multiplication of Displacement Vectors

The basic premise is this: if we multiply a vector by a scalar quantity, we transform the length of the vector by that specific factor. For example,



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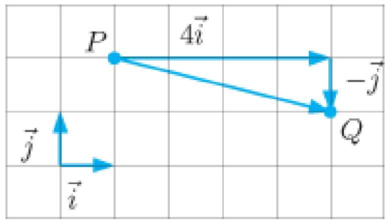
If λ is a scalar and $\vec{v}$ is a displacement vector, the **scalar multiple of** $\vec{v}$ **by** λ, written $\lambda\vec{v}$, is the displacement vector with the following properties:

- The displacement vector $\lambda\vec{v}$ is parallel to $\vec{v}$, pointing in the same direction if $\lambda > 0$ and in the opposite direction if $\lambda < 0$.
- The magnitude of $\lambda\vec{v}$ is $|\lambda|$ times the magnitude of $\vec{v}$, that is, $||\lambda\vec{v}|| = |\lambda|\,||\vec{v}||$.

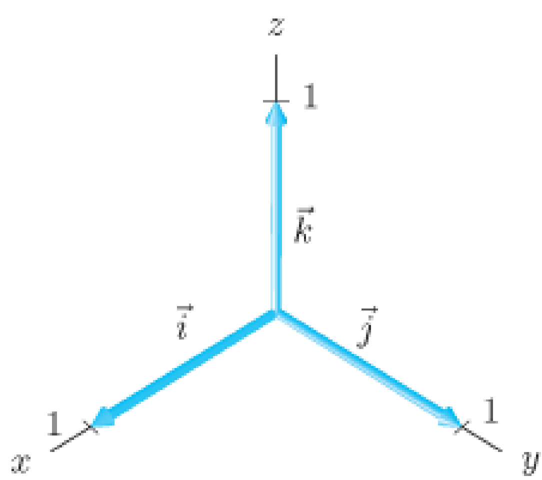
Components of Displacement Vectors: The Vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$

The vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$ are *unit vectors*, that is vectors of length 1, that point in the directions of the positive x-axis, y-axis, and z-axis, respectively. Using vector addition, we can write any vector as a sum of scalar multiples of $\vec{i}$, $\vec{j}$, and $\vec{k}$. This is especially easy to visualize in two dimensions:



In the picture above, the displacement vector from P to Q is $4\vec{i} - \vec{j}$.

Much of what we do in this course will take place in three-dimensional Euclidean space ($\mathbb{R}^3$). We will cover $\mathbb{R}^3$ in more detail once we get to chapter 12. For now, just imagine taking the xy-plane and attaching a third axis, the z-axis, in a direction perpendicular to both the x and y axis and such that the z-axis points along the length of your thumb if you curl your right hand in the direction from the x-axis to the y-axis. In such a three-dimensional coordinate system, the vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$ can be visualized as follows:



We **resolve** $\vec{v}$ into components by writing $\vec{v}$ in the form

$$\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k},$$

where v_1, v_2, v_3 are scalars. We call $v_1\vec{i}$, $v_2\vec{k}$, and $v_3\vec{k}$ the **components** of $\vec{v}$.

An Alternative Notation for Vectors

Often you will see the vector $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ written in the form

$$\vec{v} = \langle v_1, v_2, v_3 \rangle,$$

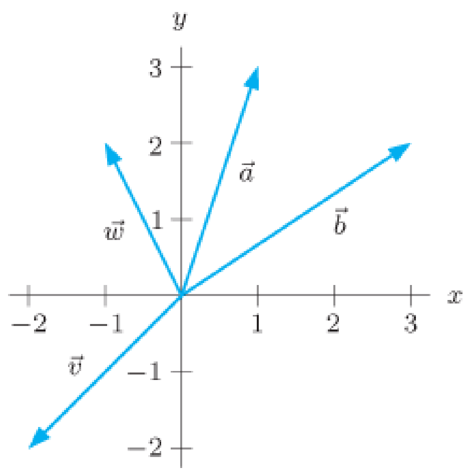
or even sometimes as

$$\vec{v} = (v_1, v_2, v_3).$$

We will often use the notation $\langle v_1, v_2, v_3 \rangle$ in this course.

Examples:

1. Resolve the vectors pictured below into components



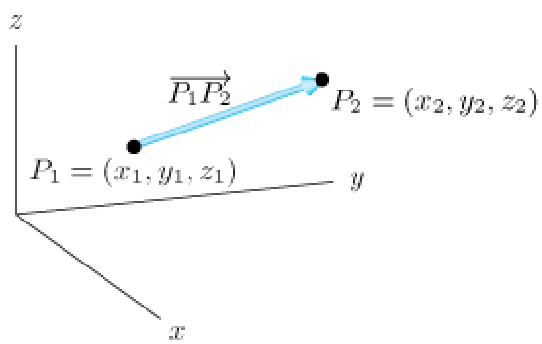
Components of Displacement Vectors

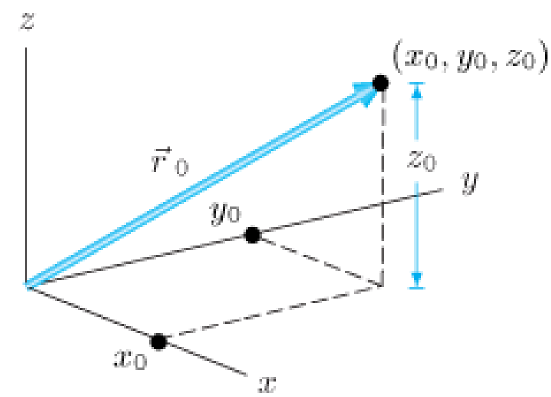
The displacement vector from the point $P_1 = (x_1, y_1, z_1)$ to the point $P_2 = (x_2, y_2, z_2)$ is given in components by

$$\vec{P_1P_2} = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k}$$

Position Vectors: Displacement of a Point from the Origin

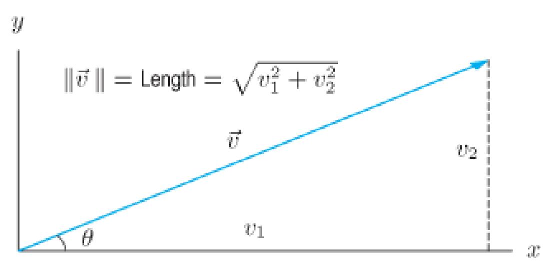
A displacement vector whose tail is at the origin is called a *position vector*. This creates a one-to-one correspondence between points (x_0, y_0, z_0) in $\mathbb{R}^3$ and vectors $\vec{r}_0$ according to $\vec{r}_0 = x_0\vec{i} + y_0\vec{j} + z_0\vec{k}$.





The Magnitude of a Vector in Components

The magnitude of a vector can be easily computed using the Pythagorean Theorem



$$\text{Magnitude of } \vec{v} = \|\vec{v}\| = \text{Length of arrow} = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

Addition and Scalar Multiplication of Vectors in Components

Addition and scalar multiplication of vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ is performed component by component. Therefore, if $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ and $\vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$, and if λ is any scalar, we have

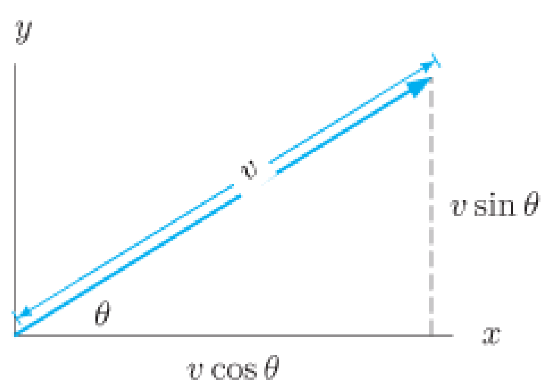
$$\vec{v} + \vec{w} = (v_1 + w_1)\vec{i} + (v_2 + w_2)\vec{j} + (v_3 + w_3)\vec{k}$$

and

$$\lambda\vec{v} = \lambda v_1\vec{i} + \lambda v_2\vec{j} + \lambda v_3\vec{k}$$

How to Resolve a Vector into Components

Suppose you have a 2-dimensional vector, $\vec{v}$, and you are given that $\vec{v}$ has length $\|\vec{v}\| = v$ and that $\vec{v}$ makes an angle of θ with the x-axis, measured counterclockwise. If $\vec{v} = v_1\vec{i} + v_2\vec{j}$, find v_1 and v_2.



Unit Vectors

Simply put, a *unit vector* is a vector whose magnitude is equal to 1. The vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$ are examples of unit vectors that we have already seen.

It is a relatively simple matter to find a unit vector that points in the same direction as an arbitrary vector $\vec{v}$. For example, suppose that $\|\vec{v}\| = 10$. Then the vector $\vec{u} = \vec{v}/10$ has length 1 and points in the same direction as $\vec{v}$.

In general, a unit vector in the direction of any given nonzero vector $\vec{v}$ is given by

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|}$$

Examples:

2. Perform the indicated operation:

$$\frac{1}{2}(2\vec{i} - \vec{j} + 3\vec{k}) + 3\left(\vec{i} - \frac{1}{6}\vec{j} + \frac{1}{2}\vec{k}\right)$$

3. (a) Draw the position vector for $\vec{v} = 5\vec{i} - 7\vec{j}$

   (b) What is $\|\vec{v}\|$?

   (c) Find the angle between $\vec{v}$ and the positive x-axis.

4. Find the unit vector in the opposite direction to $\vec{i} - \vec{j} + \vec{k}$

5. Find a vector of length 2 that points in the same direction as $\vec{i} - \vec{j} + 2\vec{k}$

6. Find the value(s) of a making $\vec{v} = 5a\vec{i} - 3\vec{j}$ parallel to $\vec{w} = a^2\vec{i} + 6\vec{j}$

7. (a) Find a unit vector from the point $P = (1, 2)$ and toward the point $Q = (4, 6)$

   (b) Find a vector of length 10 pointing in the same direction.

8. Resolve the following vectors into components

   (a) The vector in 2-space of length 2 pointing up and to the right at an angle of $\pi/4$ with the x-axis.

   (b) The vector in 3-space of length 1 lying in the xz-plane pointing upward at an angle of $\pi/6$ with the positive x-axis.