Section 17.1: Parametrized Curves

A curve in $\mathbb{R}^2$ or $\mathbb{R}^3$ can be parametrized by a series of functions of a single variable, one for each of the coordinates. In $\mathbb{R}^3$, this means that a curve can be parametrized by three functions $x = f(t)$, $y = g(t)$, and $z = h(t)$, where $t$ is allowed to vary over some domain on the real numbers. As $t$ varies, the coordinates $(x, y, z)$ trace out a curve in space.

Examples:

1. Find parametric equations for the curve $y = x^3$ in the $xy$-plane.

2. Describe in words the motion given parametrically by

$$x = \cos t, \quad y = \sin t, \quad z = t.$$

![Diagram of a parametrized curve](image)
3. Write parametric equations for the circle of radius 2, centered at the origin in the $xy$-plane.

4. Find parametric equations for the line in the direction of the vector $\vec{i} + 2\vec{j} - \vec{k}$ and through the point $(3, 0, -4)$

The previous example can be generalized to show how to obtain the parametric equations for a line that passes through any arbitrary point and points in the direction of any vector.

**PARAMETRIC EQUATIONS OF A LINE** through the point $(x_0, y_0, z_0)$ and parallel to the vector $a\vec{i} + b\vec{j} + c\vec{k}$ are

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.$$
Using Position Vectors to Write Parametrized Curves as Vector-Valued Functions

As we noted before, there is a one-to-one correspondence between points in $\mathbb{R}^2$ and $\mathbb{R}^3$ and vectors. Any point $(x, y)$ in $\mathbb{R}^2$ can be represented by the *position vector* $\vec{r} = x\vec{i} + y\vec{j}$. Similarly, any point $(x, y, z)$ in $\mathbb{R}^3$ can be represented by the position vector $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$. 
If \( x = f(t) \), \( y = g(t) \), and \( z = h(t) \) are parametric equations for a curve in \( \mathbb{R}^3 \), we can now combine these three equations into the single vector equation

\[
\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}.
\]

**Parametric Equations of a Line**

As before, consider a straight line in the direction of a vector \( \vec{v} \) which passes through the point \((x_0, y_0, z_0)\) with position vector \( \vec{r}_0 \). The idea is to start at \( \vec{r}_0 \), and move up the line, adding different multiples of \( \vec{v} \).

**PARAMETRIC EQUATION OF A LINE IN VECTOR FORM:** The line through the point with position vector \( \vec{r}_0 = x_0\vec{i} + y_0\vec{j} + z_0\vec{k} \) in the direction of the vector \( \vec{v} = a\vec{i} + b\vec{j} + c\vec{k} \) has parametric equation

\[
\vec{r}(t) = \vec{r}_0 + t\vec{v}.
\]
Examples:

5. Parametrize the line perpendicular to the plane \( z = 2x - 3y + 7 \) and through the point \((1, 1, 6)\). 

6. Parametrize the circle of radius 2 parallel to the \( xy \)-plane, centered at the point \((0, 0, 1)\), and traversed counterclockwise when viewed from below.

7. Parametrize the curve in which the plane \( z = 2 \) cuts the surface \( z = \sqrt{x^2 + y^2} \).
8. Parametrize the line through $P = (2, 5)$ and $Q = (12, 9)$ so that the points $P$ and $Q$ correspond to the parameter values $t = 0$ at $P$ and $t = 5$ at $Q$.

9. Find an equation for the plane containing the point $(2, 3, 4)$ and the line $x = 1 + 2t$, $y = 3 - t$, and $z = 4 + t$. 