Data assimilation and inverse problems – Homework Set 1

Problem 1
Is the probability measure $P(H) = 1/2, P(T) = 1/2$ a good model for a coin toss? How often do you observe $H$ (or $T$) when you toss a coin 10, 50, 100 or 500 times?

Problem 2
Let $x$ and $y$ be two random variables. Show that if $x$ and $y$ are independent, then $x$ and $y$ are uncorrelated. Find (or read about) a counter example that shows that if $x$ and $y$ are uncorrelated, they might not be independent. Show that if $x$ and $y$ are jointly Gaussian, then $x$ and $y$ are uncorrelated if and only if $x$ and $y$ are independent.

Problem 3
Let $x$ be a multivariate random variable with probability density function $p(x) \propto \exp(-F(x))$, where $F(x)$ is a quadratic function with positive definite Hessian. Show that $x$ is Gaussian with mean $\mu = \arg \min F(x)$ and that the covariance matrix is the inverse of the Hessian of $F$ evaluated at its minimizer.

Problem 4
(Box-Muller algorithm). Let $x$ and $y$ be two independent uniform random variables on $[0, 1]$. Show that $\eta_1 = \sqrt{-2\sigma^2 \log x} \cos(2\pi y)$ and $\eta_2 = \sqrt{-2\sigma^2 \log x} \sin(2\pi y)$ are independent Gaussians with mean zero and variance $\sigma^2$. 