1.1 Linear equations
- What is a linear equation?
- Matrix notation.
- What is a solution or solution set?
- Key: row-equivalence: elementary row operations do not change the solution.
- Consistent: has one or more solutions.

1.2 Row-reduction
- The cornerstone: row-reduction.
- Row-reduction can be enough to see if the system has a solution.
- Reduced row-reduction form is unique.
- Every matrix has only one reduced row echelon form.

1.3 Vector equations
- Let vector = matrix with one column.
- Know how to handle them.
- Linear combinations.
- Span \{v_1, \ldots, v_p\}.

1.4 Matrix Eq \(Ax = b\)
- \(A\mathbf{x} = x_1a_1 + x_2a_2 + \ldots + x_na_n\)
- or "row x column."
- When is \(A\mathbf{x} = b\) consistent for all \(b\)?
- How to solve \(A\mathbf{x} = b\)?
- Vector eqns = \(A\mathbf{x} = b\) = augmented matrix of linear system.
Homogeneous & Inhomogeneous Eqs

1. \( \mathbf{A} \mathbf{x} = \mathbf{0} \) = homogeneous prob.
   - Has solutions only if \( \mathbf{A} \) is a free variable
   - Always has trivial solution
   - Solves all planes through \( \mathbf{0} \)

2. \( \mathbf{A} \mathbf{x} = \mathbf{b} \) = inhomogeneous prob.
   - Any solution can be written as \( \mathbf{x} = \mathbf{p} + \mathbf{x}_h \)
   - Solutions are shifted planes

1.7 Linear Independence

3. \( \{ \mathbf{v}_1, \ldots, \mathbf{v}_p \} \) lin. indep. = \( \mathbf{x}_1 \mathbf{v}_1 + \ldots + \mathbf{x}_p \mathbf{v}_p = \mathbf{0} \)
   - Has only trivial solution

4. \( \{ \mathbf{v}_1, \ldots, \mathbf{v}_p \} \) are dependent
   - One vector is a linear combination of the rest
   - One vector is \( \mathbf{0} \).

5. If there are more vectors than elements in each vector, then they are linearly dependent.

1.8 Linear Transformations

6. Think of matrices as \( \mathbf{T} \)s!
   - \( \mathbf{T}(\mathbf{x}) = \mathbf{A} \mathbf{x} \)

7. \( \mathbf{A} \mathbf{x} \) is a linear \( \mathbf{f} \)cn.

8. Range, domain, etc can be connected to \( \mathbf{A} \mathbf{x} = \mathbf{b} \) solutions.

9. Every linear \( \mathbf{f} \)cn = \( \mathbf{A} \mathbf{x} = \mathbf{b} \)
In three planes \( \checkmark \)

\[
\begin{align*}
    x_1 + 2x_2 + x_3 &= 4 \\
    x_2 - x_3 &= 1 \\
    x_1 + 3x_2 &= 0
\end{align*}
\]

At least have one common point of intersection?

Solve system!

\[
\begin{pmatrix}
    1 & 2 & 1 & 4 \\
    0 & 1 & -1 & 1 \\
    1 & 3 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
    1 & 2 & 1 & 4 \\
    0 & 1 & -1 & 1 \\
    1 & 3 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
    1 & 2 & 1 & 4 \\
    0 & 1 & -1 & 1 \\
    0 & 0 & 0 & 2
\end{pmatrix}
\]

1) System is not consistent.
2) There is no \( x, x_1, x_3 \) such that all equal one line.
3) There is no point common to all 3 planes.

---

Check \( b \) such that

a) the system has no solution,

b) \( -c \) has a unique solution,

c) \( -c \) has many solutions

\[
\begin{align*}
    x_1 + h x_2 &= 2 \\
    4x_1 + 8x_2 &= k
\end{align*}
\]

\(
\begin{pmatrix}
    1 & h & 2 \\
    4 & 8 & k
\end{pmatrix}
\begin{pmatrix}
    1 & h & 2 \\
    0 & 8-4h & k-8
\end{pmatrix}
\)

\begin{enumerate}
    \item [a)] no solution \( \Rightarrow \) system is consistent \( \Rightarrow \) rightmost column \( 2 \) = pivot column
    \item [b)] \( h = 2 \), \( k \neq 8 \).
    \item [c)] \( k \neq 8 \) \( \Rightarrow \) 2nd column must be pivot column \( \Rightarrow h \neq 2 \\
    \begin{align*}
        x_2 &= \frac{k-8}{8-4h} \\
        x_1 &= 2 - h \cdot x_2
    \end{align*}
\end{enumerate}

\begin{enumerate}
    \item [c)] many solutions \( \Rightarrow \) we must have one free variable
    \item \( 8-4h = k-8 \)
    \item \( 1+k+4h = 16 \) \hspace{1cm} \text{e.g.} \hspace{1cm} k=4 \hspace{1cm} h=2
\end{enumerate}
\[ A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \\ 1 & -2 & 3 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \]

Why? Columns of \( A \) are linearly dependent!
\[ \text{Span } \{ \mathbf{v}_1, \mathbf{v}_2 \} = \text{C} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \]
Thus, no \( c \) such that \( \mathbf{c} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \)

Give a geometric description of \( \text{Span } \{ \mathbf{v}_1, \mathbf{v}_2 \} \) for \( \mathbf{v}_1 = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \)

Solution: For every \( \mathbf{w} \) in \( \text{Span } \{ \mathbf{v}_1, \mathbf{v}_2 \} \)
\[ \mathbf{w} = c_1 \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3c_1 - 2c_2 \\ 2c_2 + 3c_2 \end{pmatrix} \]

Thus, plane is always 2D.
Thus, \( \text{Span } \{ \mathbf{v}_1, \mathbf{v}_2 \} \) is the \( x-t \)-plane.

Let \( A \) be an \( m \times n \) matrix and let \( \mathbf{u}, \mathbf{v} \) be vectors in \( \mathbb{R}^n \) with the property \( A\mathbf{u} = 0, A\mathbf{v} = 0 \). Show that \( A(c\mathbf{u} + d\mathbf{v}) = 0 \) for every scalar constants \( c, d \).

\[ A(c\mathbf{u} + d\mathbf{v}) = A(c\mathbf{u}) + A(d\mathbf{v}) = c \cdot A\mathbf{u} + d \cdot A\mathbf{v}, \]
\[ = c \cdot 0 + d \cdot 0 = 0 \]

Are the vectors: \( \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 8 \end{pmatrix}, \begin{pmatrix} 4 \\ 8 \\ 6 \end{pmatrix} \) linearly independent?

Find values of \( h \) such that the following three vectors are linearly dependent:
\[ \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ h \end{pmatrix} \]

Find the matrix that corresponds to \( \begin{pmatrix} 1 & 3 & -1 \\ -1 & -5 & 5 \\ 4 & 7 & -5 \end{pmatrix} \)
\[ \begin{pmatrix} 1 & 3 & 1 \\ -1 & -5 & 5 \\ 4 & 7 & h \end{pmatrix} \]

Solve the system:
\[ \begin{pmatrix} 1 & 3 & 1 \\ -1 & -5 & 5 \\ 4 & 7 & h \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} \]

For linear dependence, \( h \) must have free variables

For linear dependence:
\[ \begin{pmatrix} 4 & h+4 \\ -8 & 5 \\ 16 & 10 \end{pmatrix} \]
\[ h \neq 6 \]

\[ h = 9 \]