Berezin Integrals

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1 Introduction

These notes examine a construction that is used to deal with Fermions in the context of quantum field theory and take a closer look at a simple finite dimensional example to see how it is possible to reinterpret aspects of the construction in a mathematically precise fashion that extends to the infinite dimensional examples that are of most interest in field theory. Fermions are particles in quantum mechanics that obey anti-symmetric statistics. A quantum mechanical wave functions $\psi_{ij}(x, y)$ that describes a two particle state for say an electron (which is a Fermion) is anti symmetric,

$$\psi_{ij}(x, y) = -\psi_{ji}(y, x).$$

Here x, y can be thought of as space coordinates and ij as indices that describe internal degrees of freedom (spin, charge, and etc.) The "identitities" of the individual particles in such a state are inextricably mingled. It is not possible to refer to one of the particles as "sam" and the other as "pam"; the two particle state is at best "spam". There is, incidentally, another species of particles called Bosons. These have symmetric statistics. At some fundamental level matter appears to be made up of Fermions (like electrons, quarks, and etc.) that is held together by forces mediated by Bosons (photons, gluons, and etc.). Quantum field theory attempts to combine quantum mechanics with ideas of relativity; in particular the idea that forces cannot propagate with speed greater than the speed of light leads to the notion of local interactions. It is a little complicated to describe in detail the way this plays out, but roughly speaking the interactions are formulated in terms of fields (like the electromagnetic field) which incorporate locality; the particle description of events emerges only in scattering theory. In all current field theories that incorporate Fermions there is at every point x in space-time a finite dimensional complex vector space V which describes the "local Fermionic degrees of freedom" at x. What this means at a conceptual level is obscure indeed, but mathematically the elements of V are treated as "Grassmann variables"; vectors for which there is an antisymmetric multiplication. This is simply achieved from a mathematical point of view by regarding V as a subspace of the exterior algebra,

$$\Lambda(V) = \sum_{n=0}^{\dim V} \Lambda^n(V) = \mathbf{C} \oplus V \oplus \Lambda^2(V) \oplus \cdots \oplus \Lambda^{\dim V}(V).$$

The multiplication is then given by the usual wedge product $u \wedge v$. In theories that describe "charged" Fermions the space V at a point x is spanned by vectors $\psi_j(x)$ and $\bar{\psi}_j(x)$, where the index j runs over a finite set. Interactions that are symmetric under the charge conjugation symmetry which exchanges ψ_j and $\bar{\psi}_j$, treat positively and negatively charged particles on a equal footing. "Amplitudes" for quantum processes that involve Fermions are determined by "integrals",

$$\int F(\bar{\psi}, \psi) e^{\mathcal{L}} \prod_{x,j} d\bar{\psi}_j(x) d\psi_j(x),$$

where $F(\bar{\psi},\psi)$ is a polynomial in $\bar{\psi}_k(a)$ and $\psi_l(b)$ for various values of k, l, a, b and $\mathcal{L}$ is the action,

$$\mathcal{L} = \int (D + A(x)) \bar{\psi}(x) \psi(x) dx.$$

Here D is the Dirac operator (a first order differential operator), $A(x)$ is a gauge potential (acting as a multiplication operator) and the "integration" is typically over all space-time (at least for the calculation of scattering amplitudes). In realistic theories the gauge potentials $A(x)$ are also quantum fields (Bosonic) and must be integrated over as well. This bosonic integral is more difficult to make sense of than the Fermionic integral which we will focus on.

How one tries to understand the Physics of all this is a long story (Stephen Weinberg's multivolume "The Quantum Theory of Fields" is a modern introduction to the subject); I can't do justice to the Physics but I will illustrate some of the ideas behind the Fermi integrals in a setting that does make mathematical sense. Suppose that $\{\bar{\psi}_j, \psi_j\}$is basis for a finite dimensional vector space ($j = 1, \ldots, N$). Let A_{ij} denote an $N \times N$ matrix. A finite dimensional analogue of the integral we want to understand is,

$$\int \exp(\sum A_{ij} \psi_i \wedge \bar{\psi}_j) \prod_j d\bar{\psi}_j d\psi_j. \tag{1}$$

The exponential is understood as a (finite) power series in the exterior algebra,

$$\sum_{n=0}^{N} \frac{(\sum A_{ij} \psi_i \wedge \bar{\psi}_j)^n}{n!},$$

where no terms beyond the N^{th} appear because $\Lambda^k(V) = 0$ for $k > \dim V$. The Berezin integral,

$$\int \cdot \prod_j d\bar{\psi}_j d\psi_j,$$

picks out the coefficient of the "volume element" $\psi_1 \wedge \bar{\psi}_1 \wedge \cdots \wedge \psi_N \wedge \bar{\psi}_N$ in the exterior algebra. The rules which govern such integrals are,

$$\int \psi_j d\psi_j = 1, \int 1 d\psi_j = 0, \int \bar{\psi}_j d\bar{\psi}_j = 1, \int 1 d\bar{\psi}_j = 0,$$

with anti-symmetric extension. We can evaluate the "integral" (1) by first observing that only the N^{th} power in the expansion of the exponential will contribute to the integral since none of the other terms will have a contribution from the volume element. The integral of the N^{th} power can be written (wedge products are omitted but understood to conserve space),

$$\frac{1}{N!}\int \sum_{i,j} A_{i_1 j_1} A_{i_2 j_2} \cdots A_{i_N j_N} \psi_{i_1} \bar{\psi}_{j_1} \cdots \psi_{i_N} \bar{\psi}_{j_N} d\bar{\psi}_N d\psi_N \cdots d\bar{\psi}_1 d\psi_1.$$

Here we use the fact that quadratic elements in the exterior algebra commute with one another to rearrange the factors $d\bar{\psi}_j d\psi_j$ in arbitrary order. The only terms which survive the "integration" are terms in which $i_1, i_2, \ldots, i_N$ and $j_1, j_2, \ldots, j_N$ are permutations of $1, 2, \ldots, N$. We can do the sum over the i permutations, freely reordering the quadratic elements $\psi_i \bar{\psi}_i$ to find,

$$\int \sum_{j} A_{1 j_1} A_{2 j_2} \cdots A_{N j_N} \psi_1 \bar{\psi}_{j_1} \cdots \psi_N \bar{\psi}_{j_N} d\bar{\psi}_N d\psi_N \cdots d\bar{\psi}_1 d\psi_1,$$

where the sum is over all permutations of j. It is easy to check that interchanging two indices j_k and j_l in this last formula involves an odd number of exterior algebra commutators and hence a change in sign. Thus putting the permutation j in natural order (which then produces precisely the volume element that the integral is looking for) introduces a factor $\mathrm{sgn}(j)$ (the sign of the permutation j) to give,

$$\sum_{j} \mathrm{sgn}(j) A_{1 j_1} \cdots A_{N j_N} = \det A.$$

Applying this formula (without conscience) to the field theory integral one obtains,

$$\int e^{\mathcal{L}} \prod_{x,j} d\bar{\psi}_j(x) d\psi_j(x) = \det(D + A).$$

Formulas like this were first written down by the Physicists, Matthews and Salam in the early 1950's and have led to a cottage industry trying to make sense out of determinants of differential operators. The spectrum of differential operators is such that it never makes sense to take the product of the eigenvalues (which would be the determinant) so one must be more creative (a popular extension of the determinant which works for elliptic operators on compact manifolds involves "zeta function regularization"). I won't say anything more about this aspect of things since I have in mind to explore the "local" aspect of the Berezin construction from a somewhat different angle.

In the original constructions the variable x was a point in space-time. However, in the 1970's it was discovered that in many field theory constructions one could "pass back and forth to imaginary time" where the quadratic form of relativistic physics (which is $t^2 - x^2 - y^2 - z^2$) becomes a positive definite Euclidean form. In the path integral formulations all the coordinates are then on the same footing. It is this Euclidean formulation which we will be examining. Sadly, this removes us even further from the scattering theory interpretation of the amplitudes we compute since an analytic continuation in time is required to return to the Physical regime.

In fact, to make things as simple as possible, we will suppose to begin with that x is an element of $\mathbf{Z}$, the lattice of integers (space is 0 dimensional and only (imaginary) time survives). We will model the differential operator $D + A$ by a finite difference operator, L, and we will focus on the problem of understanding how to "integrate out" all the local Fermion degrees of freedom that come from a finite part of the lattice.

We suppose that the local Fermionic degrees of freedom are given by a finite dimensional complex vector space V. Since we want to associate quadratic forms with linear operators we will also suppose that V has a distinguished non degenerate complex bilinear form $(\cdot, \cdot)$. The form $(\cdot, \cdot)$ is said to be non degenerate provided that every linear functional on V can be realized as a map,

$$V \ni v \to (v, u) \in \mathbf{C},$$

for some $u \in V$. It might help to think of this bilinear form arising from an inner product $\langle \cdot, \cdot \rangle$,

$$(u, v) = \langle Cu, v \rangle,$$

which we suppose is conjugate linear in the first slot. Here C is a conjugate linear map on V (that is, $C(\alpha v) = \bar{\alpha} C(v)$) such that $C^2 = 1$; such maps are called conjugations. For example on $V = \mathbf{C}^n$ the distinguished bilinear form,

$$(x, y) = \sum_{j=1}^{n} x_j y_j$$

arises in just this fashion from the inner product,

$$\langle x, y \rangle = \sum_{j=1}^{n} \bar{x}_j y_j,$$

with the distinguished conjugation given by complex conjugation in each coordinate. Define the vector space $V(\mathbf{Z})$ to be the space of functions $f : \mathbf{Z} \to \mathbf{V}$ with,

$$\sum_{x \in \mathbf{Z}} \langle f(x), f(x) \rangle < \infty.$$

This is, of course, just the usual ℓ^2 sequence space. Define a finite difference operator L on $V(\mathbf{Z})$ by,

$$Lf(k) = A\left(k + \frac{1}{2}\right) f(k+1) + B(k) f(k) - A\left(k - \frac{1}{2}\right)^{\tau} f(k-1).$$

Here $A(\ell) : V \to V$ is a linear map on V for each $\ell \in \mathbf{Z} + \frac{1}{2}$, the transpose A^τ is defined by,

$$(A^\tau u, v) = (u, Av),$$

for all $u, v \in V$, and finally $B(k) : V \to V$ is a linear map for each $k \in \mathbf{Z}$ with the property that $B(k)^\tau = -B(k)$. You might check that L is formally skew symmetric with respect to the bilinear form,

$$(f, g) = \sum_{k \in \mathbf{Z}} (f(k), g(k)).$$

In fact L has been contrived to be the most general range 1 finite difference operator that is skew symmetric with respect to this form. For L to be defined on all of $V(\mathbf{Z})$ it is necessary to suppose that the norms of $A(\ell)$ and $B(k)$ are uniformly bounded. The operators $A(\ell)$ have been indexed by half integers because it is marginally more natural to do so. You can imagine that the $A(\ell)$ live on the "bonds" connecting nearest neighbor integer sites–for example, the bond that connects sites 1 and 2 is naturally labeled by the midpoint $3/2$. Higher dimensional generalizations or generalizations to different lattices would naturally involve potentials defined on the bonds.

For each $k \in \mathbf{Z}$ let $\psi_j(k)$ for $j = 1, \ldots, n$ denote an orthonormal basis for V with respect to the bilinear form $(\cdot, \cdot)$. We think of $\psi_j(k)$ as an element of $V(\mathbf{Z})$ supported at k. We would like to associate an "action", $\mathcal{L}$ with L in the following manner,

$$\mathcal{L} = \sum_{j,k} L\psi_j(k) \wedge \psi_j(k).$$

This sum will not necessarily converge unless L is truncated in some fashion (which we intend to do) and you might want to check that $\mathcal{L}$ is at least formally independent of the choice of orthonormal basis for V. The desire to obtain a quadratic form, $\mathcal{L}$, from the linear operator L is the reason we need to introduce the distinguished bilinear form on V. Note that the wedge product in the definition of $\mathcal{L}$ also makes it natural to suppose that L is skew symmetric. Note that we are no longer supposing that the interaction is symmetric with respect to a charge conjugation. In Physics this situation arises for "neutral Fermions". For us it means that we will have to deal with Pfaffians rather than determinants. The resulting formalism is actually more general than the charged formalism–one can obtain the "charged" case from the "neutral case" by introducing a charge conjugation and restricting consideration to charge symmetric interactions.

To avoid worrying too much about the definition of a volume form we now suppose that V is even dimensional and that we have fixed a volume form,

$$\omega_k = \psi_1(k) \wedge \psi_2(k) \wedge \cdots \wedge \psi_n(k),$$

for $V(k)$. Then for any finite subset Ω of $\mathbf{Z}$ the volume form,

$$\omega_\Omega = \prod_{k \in \Omega} \omega_k,$$

is well defined (as a wedge product) since the volume forms ω_k commute amongst themselves. We define the Berezin functional, $\mathcal{B}_\Omega$ by,

$$\mathcal{B}_\Omega(\alpha\omega_\Omega + \text{lower order terms}) = \alpha.$$

The "lower order terms" in this expression refers to elements in the exterior algebra over $V(\Omega)$ which have lower degree than ω_Ω.

Now let Ω denote the finite "interval",

$$\Omega = \{k \in \mathbf{Z} : \ m \le k \le n\}$$

and introduce the slightly larger and slightly smaller intervals $\bar{\Omega}$ and $\underline{\Omega}$ defined by,

$$\bar{\Omega} = \{k \in \mathbf{Z} : \ m-1 \le k \le n+1\}$$

and

$$\underline{\Omega} = \{k \in \mathbf{Z} : \ m+1 \le k \le n-1\}$$

Next we define a truncated operator $L_{\bar{\Omega}}$. Let P_X denote the orthogonal projection of $V(\mathbf{Z})$ on $V(X)$. Then the matrix of $L_{\bar{\Omega}}$ relative to the splitting,

$$V(\bar{\Omega}) = V(\Omega) \oplus V(\bar{\Omega}\backslash\Omega),$$

we take to be,

$$L_{\bar{\Omega}} = \begin{bmatrix} P_\Omega L P_\Omega & P_\Omega L P_{\bar{\Omega}\backslash\Omega} \\ P_{\bar{\Omega}\backslash\Omega} L P_\Omega & 0 \end{bmatrix}.$$

We define a truncated action,

$$\mathcal{L}_{\bar{\Omega}} = \frac{1}{2} \sum_{k\in\bar{\Omega}} \sum_j L\psi_j(k) \wedge \psi_j(k).$$

Our interest in this truncated action is to determine what happens when one "integrates out" the degrees of freeom in Ω in $e^{\mathcal{L}_{\bar{\Omega}}}$. That is, we wish to calculate,

$$\mathcal{B}_\Omega\left(e^{\mathcal{L}_{\bar{\Omega}}}\right). \tag{2}$$

Evidently, the coefficient of $\omega_\Omega$in $e^{\mathcal{L}_{\bar{\Omega}}}$ is in the exterior algebra over $V(\bar{\Omega}\backslash\Omega)$ which we abreviate as $\Lambda(\bar{\Omega}\backslash\Omega)$. Since ω_Ω is an even element there is no ambiguity in identifying this element in the exterior algebra. To do the calculation in an efficient way we first recall an expansion for exponentials of quadratic elements in the exterior algebra. Suppose that L_{jk} is a skew symmetric matrix and $\{e_j\}$ is basis for an N dimensional vector space. Then,

$$\exp \frac{1}{2} \sum_{j,k} L_{j,k} e_j \wedge e_k = \sum_\sigma \mathrm{Pf}(P_\sigma L P_\sigma) e_\sigma. \tag{3}$$

In this formula the sum on the right is over all subsets $\{1 \le \sigma_1 < \sigma_2 < \cdots < \sigma_k \le N\}$ of the integers from 1 to N. This includes the empty set $\emptyset$ and we define $e_\emptyset = 1$ in the expansion with coefficient 1. The map P_σ is the projection onto the subspace spanned by the vectors $\{e_{\sigma_1}, e_{\sigma_2}, \ldots, e_{\sigma_k}\}$, the function Pf is the Pfaffian and,

$$e_\sigma = e_{\sigma_1} \wedge e_{\sigma_2} \wedge \cdots \wedge e_{\sigma_k}.$$

The Pfaffian is an algebraic square root of the determinant that is defined on skew symmetric matrices. For N even we define the Pfaffian of an $N \times N$ skew symmetic matrix L by,

$$\frac{1}{2^N N!} \left(\sum_{j,k} L_{j,k} e_j \wedge e_k \right)^{N/2} = \mathrm{Pf}(L) e_1 \wedge e_2 \wedge \cdots \wedge e_N,$$

which implies that,

$$\mathrm{Pf}(L) = \sum_\sigma \mathrm{sgn}(\sigma) L_{\sigma_1 \sigma_2} \cdots L_{\sigma_{N-1} \sigma_N}.$$

Note that N must be even for this to make sense. It is not obvious from this definition but it is none the less true that,

$$\mathrm{Pf}(L)^2 = \det L.$$

Also since a skew symmetric matrix on an odd dimensional space always has a eigenvector with eigenvalue 0 the determinant of such a matrix is always 0–it is natural that the Pfaffian is also 0 in such circumstances.

We now use (3) to evaluate (2). The only terms in $\exp \mathcal{L}_{\bar{\Omega}}$ which contribute to $\mathcal{B}_\Omega (\exp \mathcal{L}_{\bar{\Omega}})$ are those that contain the volume element from $V(\Omega)$. This means that the only terms in (3) that occur must have projections of the form $P_\Omega + P_\sigma$where P_σ is the projection on the span of $\{e_{\sigma_1}, e_{\sigma_2}, \ldots, e_{\sigma_k}\}$ and $\{e_1, e_2, \ldots, e_{2N}\}$ is an orthonormal ordered basis for $V(\bar{\Omega} \backslash \Omega)$. Note that $\bar{\Omega} \backslash \Omega$ has just two elements so the vector space $V(\bar{\Omega} \backslash \Omega)$ has dimension $2N$. One finds,

$$\mathcal{B}_\Omega (\exp \mathcal{L}_{\bar{\Omega}}) = \sum_\sigma \mathrm{Pf} \begin{bmatrix} P_\Omega L P_\Omega & P_\Omega L P_\sigma \\ P_\sigma L P_\Omega & 0 \end{bmatrix} e_\sigma.$$

The sum is as above over all ordered subsets $1 \leq \sigma_1 < \sigma_2 < \cdots < \sigma_k \leq 2N\}$. To further understand this formula we make use of the following formula for Pfaffians,

$$\mathrm{Pf} \begin{bmatrix} A & B \\ -B^\tau & 0 \end{bmatrix} = \mathrm{Pf}(-A)\mathrm{Pf}(-B^\tau A B),$$

which is valid when A is a skew symmetric invertible matrix. We suppose now (and for the rest of this note) that $P_\Omega L P_\Omega$ is invertible and we find,

$$\mathcal{B}_\Omega(\exp \mathcal{L}_{\bar{\Omega}}) = \mathrm{Pf}(-P_\Omega L P_\Omega) \sum_\sigma \mathrm{Pf}(P_\sigma L P_\Omega (P_\Omega L P_\Omega)^{-1} P_\Omega L P_\sigma) e_\sigma. \quad (4)$$

The overall factor $\mathrm{Pf}(-P_\Omega L P_\Omega)$ is interesting but it does not add much structure to the final result. We will concentrate on trying to characterize the vector on the right hand side of (4) projectively (that is, up to a constant multiplier). Define,

$$X = P_{\bar{\Omega}\backslash\Omega} L P_{\Omega} (P_{\Omega} L P_{\Omega})^{-1} P_{\Omega} L P_{\bar{\Omega}\backslash\Omega}. \tag{5}$$

Then X is a skew symmetric map on $V(\bar{\Omega}\backslash\Omega)$ and the vector we are interested in is,

$$\sum_{\sigma} \mathrm{Pf}(P_\sigma X P_\sigma) e_\sigma = \exp \frac{1}{2} \sum_{j=1}^{2N} X e_j \wedge e_j, \tag{6}$$

where we used (3) again. Thus the vector we are interested in is the exponential of a quadratic form in $\Lambda(\bar{\Omega}\backslash\Omega)$. Such vectors can be understood as "vacuum vectors" for representations of a Clifford algebra. We turn next to this topic.

It is useful to introduce creation operators, $c(u)$, acting on $\Lambda(\bar{\Omega}\backslash\Omega)$ by,

$$c(u)v = u \wedge v \text{ for } u \in V(\bar{\Omega}\backslash\Omega) \text{ and } v \in \Lambda(\bar{\Omega}\backslash\Omega).$$

Let $V(\bar{\Omega}\backslash\Omega)^*$ denote the complex dual of $V(\bar{\Omega}\backslash\Omega)$ and for $w \in V(\bar{\Omega}\backslash\Omega)^*$ we define an annihilation operator, $a(w)$, acting on $\Lambda(\bar{\Omega}\backslash\Omega)$ by,

$$a(w)e_\sigma = \sum_{j=1}^{k} (-1)^{j-1} w(e_{\sigma_j}) e_{\sigma\backslash\sigma_j},$$

where $w(e)$ is just the value of the linear functional w on the vector e and,

$$e_{\sigma\backslash\sigma_j} = e_{\sigma_1} \wedge \cdots \wedge \hat{e}_{\sigma_j} \wedge \cdots \wedge e_{\sigma_k},$$

where $\hat{e}_{\sigma_j}$ means "omit the factor e_{σ_j} in the product". The creation operators anti-commute amongst themselves,

$$c(u_1)c(u_2) + c(u_2)c(u_1) = 0,$$

as do the annihilation operators,

$$a(w_1)a(w_2) + a(w_2)a(w_1) = 0.$$

Together they have a scalar anti-commutator,

$$a(w)c(u) + c(u)a(w) = w(u)1.$$

In quantum field theory where particles are created and destroyed it is not surprising that creation and annihilation operators play an important role. Free fields are expressed as sums of creation and annihilation operators. The representation of the Clifford algebra we are interested in will also be in terms of creation and annihilation operators. Define,

$$Q = V(\bar{\Omega}\backslash\Omega), \text{ and } P = V(\Omega\backslash\underline{\Omega}),$$

$$\partial\Omega = \bar{\Omega}\backslash\underline{\Omega}.$$

so that,

$$V(\partial\Omega) = P \oplus Q.$$

The Clifford algebra we wish to discuss is generated by the vector space $V(\partial\Omega)$ together with a symmetric bilinear form on $V(\partial\Omega)$ that we now introduce. The truncated operator $P_\Omega L$ is not skew symmetric. A symmetric operator on $V(\bar{\Omega})$ which measures the failure of skew symmetry is,

$$(P_\Omega Lf, g) + (f, P_\Omega Lg) = ((P_\Omega L - LP_\Omega)f, g) = (Sf, g).$$

Using the fact that L is a range 1 finite difference operator one sees at once that,

$$S = P_\Omega L - LP_\Omega,$$

is a symmetric linear transformation on $V(\partial\Omega)$. Using the fact that L is a finite difference operator with unit range one sees that $S : Q \to P$ and $S : P \to Q$. To avoid technical difficulties we now suppose that,

$$S : V(\partial\Omega) \to V(\partial\Omega),$$

is non-singular (i.e., invertible). This implies that the symmetric form,

$$(f, g)_S = (Sf, g),$$

is non-degenerate on $V(\partial\Omega)$. This makes it possible to identify P with the dual Q^* via the symmetric form $(\cdot, \cdot)_S$.

The Clifford algebra over $V(\partial\Omega)$ associated with the symmetric form $(\cdot, \cdot)_S$ is the associative algebra with unit e generated by the elements of $V(\partial\Omega)$ subject to the relations,

$$fg + gf = (f, g)_S e.$$

The representation of this Clifford algebra we are interested in is,

$$V(\partial\Omega) \ni f = f_P \oplus f_Q \to c(Sf_P) + a(f_Q) = \Psi(f),$$

where we understand f_Q as an element of Q^* via the inner product $(f_Q, \cdot)$. It is a short exercise to show that,

$$\Psi(f)\Psi(g) + \Psi(g)\Psi(f) = (f, g)_S 1. \tag{7}$$

Thus $f \to \Psi(f)$ is a representation of the Clifford algebra over $V(\partial\Omega)$ associated with the symmetric form $(\cdot, \cdot)_S$. Our interest in this representation of the Clifford algebra has to do with the following result.

***Theorem** 1. The vector*

$$v_\Omega = \mathcal{B}_\Omega \left(\exp \mathcal{L}_{\bar{\Omega}}\right),$$

is projectively characterized (that is up to a constant multiplier) as the unique vector in $\Lambda(Q)$ that is annihilated by all elements $\Psi(f)$ where f is *the boundary value in $V(\partial\Omega)$ of a function $F \in V(\bar{\Omega})$ in the null space of $P_\Omega L$. That is,*

$$\Psi(f)v_\Omega = 0$$

whenever, $f = F|_{\partial\Omega}$, and,

$$P_\Omega LF = 0.$$

The proof of this result is not complicated and the first object of these notes once we are past the introduction will be to introduce the machinery to prove this. The reason that it is interesting is that (7) makes sense even for cases where $V(\partial\Omega)$ is infinite dimensional. The space $N_\Omega(L)$ of boundary values of elements in the null space of $P_\Omega L$ is a maximal isotropic subspace of $V(\partial\Omega)$ (this is a subspace on which the form S vanishes identically and it one that is maximal with respect to this property). Even in infinite dimensions it is understood how to construct a representation of the Clifford algebra in which a maximal isotropic subspace kills a distinguished vacuum vector. Thus we've found a reinterpretation of the notion of integrating out the local Grassmann degrees of freedom which generalizes the naive calculations in finite dimensions. There is still a lot left to do, however. One obvious construction that we would like to understand involves an interval Ω which is written as the disjoint union $\Omega_1 \cup \Omega_2$ of two contiguous intervals. In some sense it should be true that,

$$v_\Omega = \mathcal{B}_{\Omega_1\cap\Omega_2}(v_{\Omega_1} \wedge v_{\Omega_2} \exp \mathcal{L}_{\bar{\Omega}_1\cap\bar{\Omega}_2}).$$

Making sense of this in Clifford algebra context is important and is the first step in formulating a transfer matrix formalism in one dimension. Infinite volume correlations should result from a dual pairing between states obtained by integrating out degrees of freedom inside Ω and states obtained by integrating out the degrees of freedom in the complement $\mathbf{Z}\backslash\Omega$. This should have an expression in the Clifford algebra formalism. What happens in higher dimensions (2,3,4 and etc.) is worth pursuing and might be simpler to explore on lattices that are not forced to be rectangular. Finally it is of interest to look at the continuum limit–a limit in which the lattice spacing tends to 0. This is of interest in understanding the construction on manifolds where the vector space V is replaced by a vector bundle. Just how far you want to pursue these ideas for the RTG will depend on your interests and time. For the present I will turn to material about Clifford algebras and Pfaffians that is of use in understanding the proof of the last theorem.

2 Clifford Algebras

Clifford Algebras were introduced in the 19th century by William Clifford for reasons that I've never been clear about. Clifford is also well known for postulating that matter is a manifestation of the curvature of space about 40 years before Einstein's space-time curvature account of gravitation. Clifford died at age 34 of tuberculosis not long after writing about the curvature of space and its connection with matter.

About 1930 Dirac produced a resurgence of interest in Clifford algebras when he used them to define the Dirac equation for relativistic electrons (and positrons). Not long after this Brauer and Weyl generalized Dirac's idea to show how Clifford Algebras could be used to give an explicit description of a representation of the orthogonal group that had been known on the infinitesimal level (i.e., for the Lie algebra) but for which there was no "natural" model known at the time. This representation is now called the spin representation of the orthogonal group–I think the name comes from the fact that "spin" is an attribute of the electrons and positrons that emerges in the Dirac theory. Incidentally, in the spin representation, the Clifford algebra is generated by the representatives of reflections in the orthogonal group.

We will concentrate on just one aspect of the representation theory for Clifford algebras that plays a role in the proof of the theorem above. Suppose that V is an even dimensional complex vector space of dimension N and that $(\cdot,\cdot)$ is a distinguished non degenerate complex bilinear form on V. We will say that V is a complex orthogonal space. The Clifford algebra of V is the associative algebra with unit e generated by the elements of V subject to the relations,

$$xy + yx = (x, y)e \text{ for x, y} \in \text{V}.$$

The representations of the Clifford algebra that are of most interest to us are parametrized by *isotropic* splittings of V. A subspace W of V is said to be isotropic if for each pair $x, y \in M$ we have $(x, y) = 0$. A direct sum decomposition,

$$V = V_+ \oplus V_-,$$

is said to be an isotropic splitting if $V_\pm$ are both isotropic subspaces. Such splittings are also referred to as *polarizations* of V.

Now we describe the representation of the Clifford algebra associated with this splitting. Define,

$$\Psi(x) = c(x_+) + a(x_-),$$

where $x = x_+ + x_-$ is the splitting of x in $V_+ \oplus V_-$ and the creation and annihilation operators both act on,

$$\Lambda(V_+) := \mathbf{C} \oplus V_+ \oplus \Lambda^2(V_+) \oplus \cdots \oplus \Lambda^{N/2}(V_+).$$

The vector x_- is identified with an element of the dual V_+^* via the inner product, $(x_-, \cdot)$. The map,

$$x \to \Psi(x),$$

is a representation of the Clifford algebra. This representation is called the Fock representation of $\text{Cliff}(V)$ associated with the polarization $V_+ \oplus V_-$. The Russian physicist Fock introduced such a construction for free quantum fields. In that application the vector space V is infinite dimensional–it is one of the irreducible projective unitary representations of the Poincare' group.

As a first exercise in getting acquainted with these constructions:

Exercise 1

- *Show that V_+ and V_- have the same dimension (i.e., $N/2$) by showing that $V_+^* \simeq V_-$.*
- *Verify the anti-commutation relations for creation and annihilation operators from the previous section.*
- *Check that $\Psi(x)$ just described does give a representation of the Clifford algebra relations.*
- *A harder exercise is to show that this representation is irreducible. That is, the only subspaces of $\Lambda(V_+)$ that are invariant under the action of $\Psi(x)$ for all $x \in V$ are the 0 subspace and all of $\Lambda(V_+)$.*

You might wonder about the existence of isotropic splittings. There are lots. It is not hard to show that V has a basis,

$$\{v_1, \ldots, v_N\}$$

which is orthonormal, $(v_i, v_j) = \delta_{ij}$, with respect to the complex bilinear form $(\cdot,\cdot)$(a suitable modification of the Gram Schmidt process works). Check that if $V_\pm$ is the complex linear span of $v_j \pm i v_{j+N/2}$ for $j = 1, \ldots, N/2$ then $V_+ \oplus V_-$ is a polarization of V.

Another important feature of this construction is that the Fock representations associated with different polarizations of V are equivalent. If Ψ and Ψ' are represention of the Clifford relations associated with polarizations $V_+ \oplus V_-$ and $V'_+ \oplus V'_-$ then there exists an invertible linear map,

$$U : \Lambda(V_+) \to \Lambda(V'_+),$$

so that,

$$U\Psi(x) = \Psi'(x)U \text{ for } x \in V.$$

In fact, it is possible to introduce Hermitian symmetric inner products on $\Lambda(V_+)$ and $\Lambda(V'_+)$ and to choose the map U so that it is unitary. I haven't introduced this unitary structure since I think it is simpler to proceed without it at first. Later, when a quantum mechanical interpretation is desired for our constructions it will be important to describe this Hermitian structure.

The construction of Fock representations goes through in infinite dimensions with suitable restrictions on the continuity of the polarization. However, it is no longer the case that the Fock representations associated to different polarizations are unitarily equivalent. If Q is the operator on V which is the identity on V_+ and minus the identity on V_- then the Fock representations associated with Q and Q' are equivalent iff $Q - Q'$ is Schmidt class (a compact operator with a square summable sequence of eigenvalues). These are interesting matters but not essential for the proof of theorem 1 above. This infinite dimensional construction is, however, an important motivating concern for us. The theorem we are proving translates a construction (the Berezin integral) from a setting which has no obvious infinite dimensional generalization to a representation of Clifford algebras which does have a natural infinite dimensional formulation.

3 Pfaffians

There exists an algebraic square root for the determinant of a skew symmetric matrix known as a Pfaffian. We introduced this notion above and you can find a more detailed discussion of the properties of Pfaffians in the appendix to a book on the Ising model that I will make available in PDF form on my web site. There is a similar discussion of Pfaffians in the book "Loop Groups" by G. Segal and A. Pressley. I will rely on you to fill in backround by reading and here I will be brief and just show you a standard reduction formula that allows one to calculate Pfaffians of low dimensional matrices. First, the 2×2 case,

$$\mathrm{Pf}\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} = a.$$

In general if A is an $n \times n$ matrix then,

$$\mathrm{Pf}A = \sum_{k=1}^{n} (-1)^{j+k+1} a_{jk} \mathrm{Pf}A_{jk},$$

where a_{jk} is the jk matrix element of A, and A_{jk} is the $(n-2) \times (n-2)$ skew symmetric matrix obtained from A by deleting the j and k rows and the j and k columns. Note that this is a bit analogous to expanding a determinant by minors of the j^{th} row.

Pfaffians of odd dimensional matrices are always 0 (a reflection of the fact that a skew symmetric matrix in odd dimensions always has 0 as an eigenvalue). As an exercise use the reduction formula to compute the Pfaffian of the most general 4×4 skew symmetric matrix:

$$\mathrm{Pf}\begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix}.$$

Now I will finish the proof of theorem 1. Recall that the vacuum vector we are trying to characterize is,

$$v_\Omega = \exp\frac{1}{2}\sum_{j=1}^{2N} Xe_j \wedge e_j = \exp\frac{1}{2}\sum_{j=1}^{2N} c(Xe_j)c(e_j) \cdot |0\rangle,$$

where $|0\rangle = 1 \oplus 0 \oplus 0 \cdots \oplus 0 \in \Lambda(Q)$ and $c(u)$ is the creation operator associated with $u \in Q$. Next observe that for finite dimensional linear maps A and B acting on the same vector space,

$$e^{\lambda A} B e^{-\lambda A} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathrm{ad}^n(A)B, \tag{8}$$

where, $\mathrm{ad}^0(A)B = B$, $\mathrm{ad}^1(A)B = AB - BA = [A, B]$ and,

$$\mathrm{ad}^2(A)B = [A, [A, B]],$$

is the two fold iterated commutator of B with A. In general,

$$\mathrm{ad}^n(A)B = [A, [A, \cdots [A, B]],$$

is the n-fold iterated commutator. You should verify that the right hand side of 8 is just the Taylor series expansion of the left hand side about $\lambda = 0$. Now suppose that $f \in Q$. Use 8 to show that,

$$\exp \frac{1}{2} \sum_{j=1}^{2N} c(Xe_j)c(e_j) \cdot a(f) \cdot \exp -\frac{1}{2} \sum_{j=1}^{2N} c(Xe_j)c(e_j) = c(Xf) + a(f).$$

From this it follows that,

$$(c(Xf) + a(f))\, v_\Omega = \exp \frac{1}{2} \sum_{j=1}^{2N} c(Xe_j)c(e_j) \cdot a(f) \cdot |0\rangle = 0, \text{ for } f \in Q, \tag{9}$$

since $a(f)|0\rangle = 0$. Now recall the representation of the Clifford algebra for $(V_{\partial\Omega}, S)$ given by,

$$\Psi(f) = c(Sf_P) + a(f_Q).$$

Suppose that $f \in V_{\partial\Omega}$ is the boundary value of a function $F \in V(\bar{\Omega})$ (i.e., $f = F|_{\partial\Omega}$)with,

$$P_\Omega LF = 0. \tag{10}$$

Write $N_{\partial\Omega}$ for the subspace of $V_{\partial\Omega}$ which consists of boundary values of functions $F \in V(\bar{\Omega})$ which satisfy 10. Now suppose that F satisfies 10 and write,

$$f = F|_{\partial\Omega} = f_P + f_Q,$$

and,

$$F = F_\Omega + f_Q.$$

The equation which F must satisfy becomes,

$$P_\Omega L P_\Omega F_\Omega + P_\Omega L f_Q = 0,$$

from which it follows that,

$$F_\Omega = -(P_\Omega L P_\Omega)^{-1} P_\Omega L f_Q. \tag{11}$$

Recall that we suppose $P_\Omega L P_\Omega$ is invertible. Now apply $P_{\bar{\Omega}\setminus\Omega} S$ to both sides of this last equality and recall the definitions for S and X to find,

$$Sf_P = Xf_Q.$$

Thus we see that for $f \in N_{\partial\Omega}$ we have,

$$\Psi(f) = c(Sf_P) + a(f_Q) = c(Xf_Q) + a(f_Q).$$

Consulting 9 we see that,

$$\Psi(f)v_{\partial\Omega} = 0 \text{ for } f \in N_{\partial\Omega}. \tag{12}$$

It remains to check that $v_{\partial\Omega}$ is projectively characterized by this condition. This turns out to be equivalent to showing that $N_{\partial\Omega}$ is a maximal isotropic subspace of $(V_{\partial\Omega}, S)$. To see this suppose that $f, g \in N_{\partial\Omega}$ are boundary values of F, G satisfying 10. Then,

$$(Sf, g) = (P_\Omega LF, G) + (F, P_\Omega LG) = 0.$$

This shows that $N_{\partial\Omega}$ is isotropic. Because S interchanges Q and P and the subspaces are orthogonal to one another with respect to the symmetric form $(\cdot,\cdot)$, it follows that Q and P are complementary isotropic subspaces for the symmetric form $(\cdot,\cdot)_S$. Since S is supposed non-degenerate this means that both Q and P are maximal isotropic subspaces. However, equation 11 shows that $N_{\partial\Omega}$ is a graph over Q – thus it has the same dimension as Q and so must be a maximal isotropic subspace. To finish the proof that 12 projectively characterizes $v_{\partial\Omega}$ first show that the vector,

$$|0\rangle = 1 \oplus 0 \oplus \cdots \oplus 0 \in \Lambda(V_+),$$

is projectively characterized by the condition,

$$\Psi(f)|0\rangle = 0, \text{ for } f \in V_-,$$

in the Fock representation of the Clifford relations, Ψ, associated with the polarization $V_+ \oplus V_-$. Then use the equivalence of representations.