EQUILIBRIUM BIDDING STRATEGIES IN COMMON-VALUE SEALED-BID AUCTIONS

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Abstract. We consider sealed-bid auctions of an item with unknown, but common value to all bidders, and assume that each bidder has an estimate of the item's value. Formulas are developed for the expected profit of bidders under various bidding strategies in both first-price and second-price auctions. We derive unique Nash equilibrium strategies for bidders in both types of auctions and analyze their effects on both sellers and bidders. It is shown that both first-price and second-price auctions return less than the item's actual value to the seller. However, this loss is much less for second-price auctions. In particular we see that the Revenue Equivalence Theorem does not apply to our type of common-value auctions.

Auction Models. Prompted by the huge value of goods, services, and rights that exchange hands in auctions, a considerable literature on auction theory has been developed. See Klemperer (1999), Klemperer (2004), and McAfee & McMillan (1987). Much of this work has centered around two distinct auction models, ***independent-private-value*** auctions and ***common-value*** auctions. The former has been extensively analyzed, culminating in the Revenue Equivalence Theorem. Much less seems to be known about the subject of our investigation, the common-value model. As a specific application of this type of auction, the reader may wish to consider government auctions of mineral rights on given tracts of land or sea bottom.

Assumptions. An item having an unknown, but common value to each of n bidders is to be sold in a sealed-bid auction. Each bidder, i, for $i = 1, \ldots, n$, receives a ***signal***, s_i, that estimates the value, v, of the item. We suppose that all bidders have equal abilities to estimate the value of the item and that, on average, they are correct. Mathematically, we assume that the i^{th} bidder's signal is an observation of a random variable S_i with $E(S_i) = v$, and that the S_i's are independent and identically distributed. We let $R_i = S_i - v$ be the error random variables, and note that each of the R_i's, for $i = 1, \ldots, n$, has some common distribution, R. This is often described by the statement that the bidders are symmetric. Call R the common ***signal error*** random variable, and note that $m_R = 0$.

Let R have standard deviation, s, and denote its density and distribution functions by f_s and F_s, respectively. We will often illustrate our work with examples where R is either normal or uniform. However, unless specifically noted to the contrary, all of our results apply to any sufficiently smooth continuous distribution.

The random variables S_i are assumed to be statistically independent. It is, however, clear that the bidders' estimated values are ***not*** independent. Since there is a common value for the item being auctioned, knowledge of other bidders' signals would generally cause one to revise his or her estimate of the item's value.

Finally, we assume that all bidders are risk neutral and that the payment made by the winning bidder depends only upon the original set of bids that were submitted.

Setting Up The Model. Bidders and sellers have very different roles in auctions. The seller can decide what type of auction will be conducted, possibly including a reserve price below which the item will not be sold. However, he or she has no control over the bidding. A bidder is free to determine its own bid, but cannot specify the auction rules. We will consider our auction model from both the seller's and the bidders' perspectives, starting with that of the bidders.

Although it is common to consider strategy for single bidders, we will expand our analysis to include ***bidding rings***. In particular, suppose that the n bidders are partitioned into Bidding Ring 1 and Bidding Ring 2, with m and $n - m$ members, respectively. Note that in the analysis of a single bidder vis-à-vis all others, we have $m = 1$.

Historical experience (often disastrous) indicates that it is foolish for all bidders to submit their signals as their bids. The realization has evolved that signals must be reduced before bidding. We suppose that all bidders in Ring 1 bid their signals minus an amount c, and all bidders in Ring 2 bid their signals minus a fixed amount d. A winning bidder experiences a positive profit if it pays less than the common value of the item being sold, and experiences a negative profit (loss) if it pays more than the value. A non-winning bidder has a profit of 0. The quality of a bidding strategy is measured by the expected value of a bidder's profit. This is a function of c, d, n, m, s, the distribution of R, and the type of auction that is conducted.

Since the ranking of signals, and hence their errors, determines the outcome of an auction, we introduce notation for order statistics. Let E_k be the k^{th} largest error in a set of signals for the n bidders, where E_1 is the smallest and E_n is the largest. The expected value of E_k is given by

$$L(n, s, k) = \int_{-\infty}^{\infty} x \cdot \frac{n!}{(k-1)! \cdot (n-k)!} \cdot f_s(x) \cdot F_s^{k-1}(x) \cdot \left(1 - F_s(x)\right)^{n-k} \, dx.$$

FIRST-PRICE SEALED-BID AUCTIONS

Basic Bidding Strategies. Each bidder submits a secret bid to the seller. The bidder with the highest bid wins the item and pays the amount of its bid. The profit of the winning bidder is positive if its signal was lower than the value. That is, its signal error was negative. Likewise its profit is negative if its signal error was positive. Specifically, the winner's profit is the negative of its signal error. We let $V_1(c, d, n, m, s)$ be the expected value of profit for a bidder in Ring 1. The subscript 1 is used to denote a first-price sealed-bid auction.

If all bidders submitted their signals as bids, the bidder with the largest signal error would win and could expect to over pay by $L(n, s, n)$. This quantity, which does not depend upon m, is known as the ***winner's curse*** and is denoted by $C_1(n, s)$. Since all bidders have an equal chance of getting the largest error, the probability of winning for any one bidder is $1/n$. Thus,

$$V_1(0, 0, n, m, s) = -C_1(n, s)/n.$$

This expected loss could be eliminated by the ***first bidding strategy***, where each bidder submits a bid that is its signal minus $C_1(n, s)$. With this strategy, $V_1(C_1(n, s), C_1(n, s), n, m, s) = 0$ and, on average, the item is sold for its actual value.

Notice that, in a first-price sealed-bid auction, there is no advantage in bidding more than the second highest bid. This suggests a ***second bidding strategy***, where each bidder reduces its signal by $C_1(n, s) + L(n, s, n) - L(n, s, n-1)$. With this strategy, the winning bidder would have an expected profit of $L(n, s, n) - L(n, s, n-1)$. We denote this quantity by $B_1(n, s)$ and call it the ***winner's blessing*** in a first-price sealed-bid auction.

$$V_1(C_1(n, s) + B_1(n, s), C_1(n, s) + B_1(n, s), n, m, s) = B_1(n, s)/n$$

For fixed n, both the curse and blessing are directly proportional to s, with the constants of proportionality depending upon n.

$$C_1(n,s) = s \cdot C(n,1) \qquad\qquad B_1(n,s) = n \cdot s \cdot \left(C_1(n,1) - C_1(n-1,1)\right)$$

The curse and blessing have simple formulas for specific error distributions and numbers of bidders.

R **normal:** $C_1(2,s) = \dfrac{s}{\sqrt{p}}$ and $B_1(2,s) = \dfrac{2 \cdot s}{\sqrt{p}}$.

R **uniform on $[-\sqrt{3} \cdot s, \sqrt{3} \cdot s]$:** $C_1(n,s) = \dfrac{n-1}{n+1} \cdot \sqrt{3} \cdot s$ and $B_1(n,s) = \dfrac{2}{n+1} \cdot \sqrt{3} \cdot s$

for all n.

Any knowledgeable bidder can make the above calculations and predict the expected results if all bidders follow either the first or second bidding strategies. Now we consider bidding as a game. Is the first bidding strategy stable, or would it be better for bidders in Ring 1 to deviate from the plan of subtracting only the winner's curse? The answer requires a way to evaluate $V_1(c, d, n, m, s)$.

Lemma 1. Suppose that, under our common-value model, in a first-price sealed-bid auction with n bidders, the m bidders in Ring 1 bid their signals minus an amount c and the $n - m$ bidders in Ring 2 bid their signals minus an amount d. For any distribution of signal errors having a standard deviation of s, with density f_s, and distribution F_s, we have the following expected value for each Ring 1 bidder.

$$V_1(c,d,n,m,s) = -\int_{-\infty}^{\infty} (x-c) \cdot F_s^{m-1}(x) \cdot f_s(x) \cdot F_s^{n-m}(x-c+d)\, dx. \qquad (1)$$

Proof. Let the random variables $R_1, R_2, \ldots, R_m$ be the signal errors of the m Ring 1 bidders and let $R_{m+1}, R_{m+2}, \ldots, R_n$ be the signal errors of the $n - m$ Ring 2 bidders. If $M_1 = Max(R_1, R_2, \ldots, R_m)$ and $M_2 = Max(R_{m+1}, R_{m+2}, \ldots, R_n)$, then we have the following.

$$F_{M_1}(x) = F_s^m(x) \qquad f_{M_1}(x) = m \cdot f_s(x) \cdot F_s^{m-1}(x)$$

$$F_{M_2}(x) = F_s^{n-m}(x) \qquad f_{M_2}(x) = (n-m) \cdot f_s(x) \cdot F_s^{n-m-1}(x)$$

Hence,

$$V_1(c,d,n,m,s) = -\frac{1}{m} \cdot \int_{-\infty}^{\infty} (x-c) \cdot \left[(n-m) \cdot \int_{-\infty}^{x-c+d} f_s(y) \cdot F_s^{n-m-1}(y)\, dy \right] \cdot m \cdot f_s(x) \cdot F_s^{m-1}(x)\, dx$$

$$= -\int_{-\infty}^{\infty} (x-c) \cdot \left[F_s^{n-m}(x-c+d) \right] \cdot f_s(x) \cdot F_s^{m-1}(x)\, dx$$

$$= -\int_{-\infty}^{\infty} (x-c) \cdot F_s^{m-1}(x) \cdot f_s(x) \cdot F_s^{n-m}(x-c+d)\, dx.$$

This completes the derivation of *Equation 1*.

From the point of view of Ring 1 bidders, we can think of d, n, m, and s as fixed, and consider expected value as a function of our reduction, c. For very large values of c, a Ring 1 bidder will make extremely low bids. Hence, it will have a very small probability of winning, but will, on average, make a large positive profit if it does win. We could expect to have $V_1(c, d, n, m, s)$ approach 0 from the positive side, as c increases without bound. Looking in the other direction, small (possibly negative) values of c will lead to very high bids, with a strong probability of winning the auction. However, the high bids will result in large losses. It is reasonable to expect that $V_1(c, d, n, m, s)$ will approach $-\infty$ as c decreases without bound.

It turns out that all plots of $V_1(c, d, n, m, s)$ for any error distribution and fixed d, n, m, and s have similar shapes. We will illustrate this with a normal error distribution and a reduction of $C_1(n, s)$ by all Ring 2 bidders.

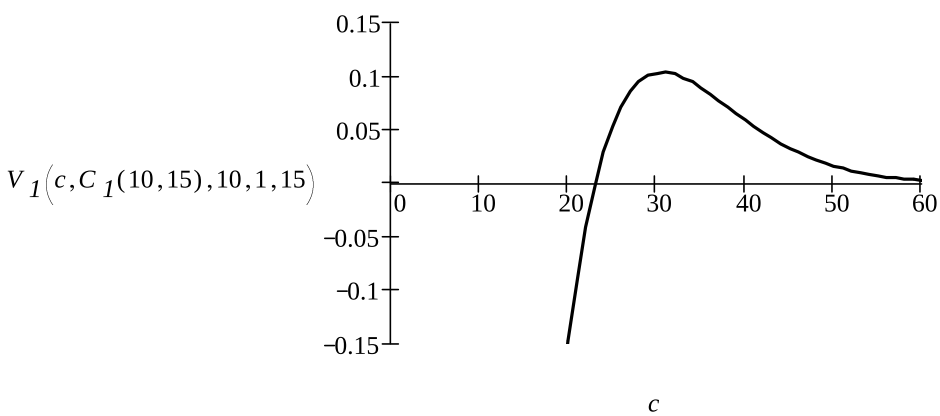

Figure 1. Normal Error Distribution.

Figure 1 shows that there is an optimum signal reduction of approximately 30.6 for Ring 1 bidders. Since, in this example, $C_1(10, 15) \cong 23.1$, we see that the first bidding strategy is not stable. Acting in its best interest, a Ring 1 bidder should subtract much more than $C_1(10, 15)$ from its signal.

As a second example, suppose that the errors are normally distributed and that all Ring 2 bidders reduce their signals by $C_1(10, 15) + B_1(10, 15)$. *Figure 2* shows that the optimal strategy for Ring 1 bidder's is to reduce their signals by approximately 23.6.

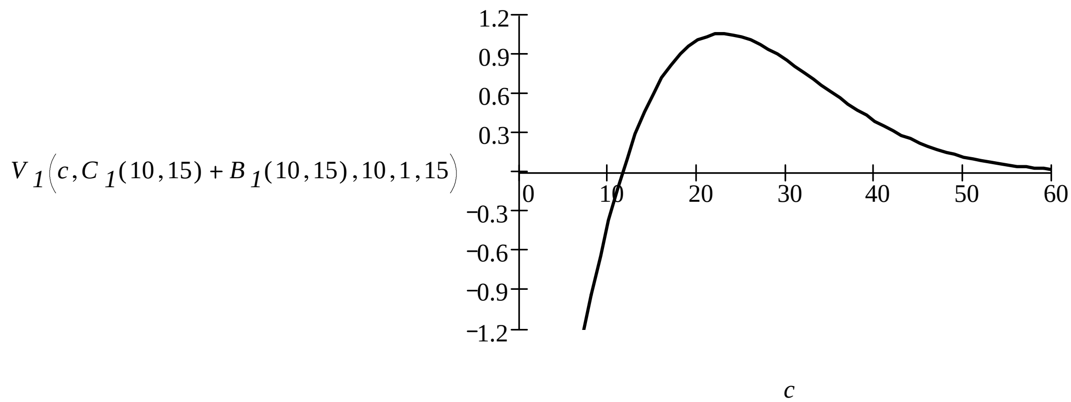

Figure 2. Normal Error Distribution.

Computation shows that, in this setting, $C_1(10, 15) + B_1(10, 15) \cong 31.1$. Hence the second bidding strategy is not stable. If Ring 2 bidders remove the winner's curse plus the winner's blessing, then a Ring 1 bidder should remove a much smaller amount from its signal.

In general, let $S_1(d, n, m, s)$, be the reduction in Ring 1 bidders' signals that maximizes their expected value, given that all Ring 2 bidders reduce their signals by d. In our examples, $S_1(23.1, n, m, s) \cong 30.6$ and $S_1(31.1, n, m, s) \cong 23.6$.

Equilibrium. Computation shows that large values of d produce $S_1(d, n, m, s) < d$ and that small values of d produce $S_1(d, n, m, s) > d$. This result holds for all error distributions and all values of n, m, and s. There is an obvious suggestion of a fixed point for $S_1(d, n, m, s)$. This, in fact, does occur. A plot of $S_1(d, n, m, s)$ against d in the setting of our last examples is shown in *Figure 3*.

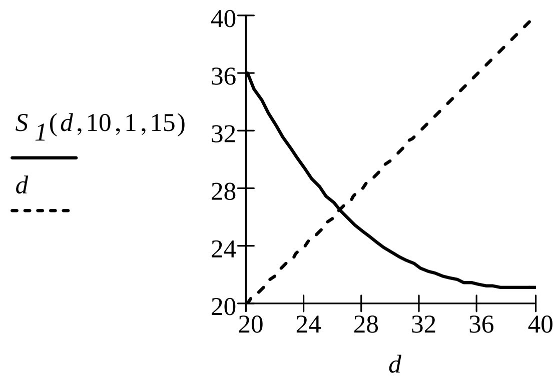

Figure 3. Normal Error Distribution.

A solution of $S_1(d, n, m, s) = d$ yields a ***Nash equilibrium bidding strategy***. ***If all bidders bid their signals reduced by this value, then no group of m bidders can increase their expected values by altering the strategy.***

Theorem 1. Consider our common-value model, for a first-price sealed-bid auction with n bidders. For any distribution of signal errors having a standard deviation of s, with density f_s, and distribution F_s, there exists a unique value $E_1(n, m, s)$ such that $S_1(E_1(n, m, s), n, m, s) = E_1(n, m, s)$. Moreover,

$$E_1(n,m,s) = \frac{\dfrac{1}{n\cdot(n-m)} + \displaystyle\int_{-\infty}^{\infty} x\cdot F_s^{n-2}(x)\cdot f_s^2(x)\,dx}{\displaystyle\int_{-\infty}^{\infty} F_s^{n-2}(x)\cdot f_s^2(x)\,dx}. \qquad (2)$$

Proof. There is an ingenious way to find this fixed point. We denote the partial derivative of $V_1(c, d, n, m, s)$ with respect to c by $D_1(c, d, n, m, s)$,

$$D_1(c,d,n,m,s) = \frac{\partial}{\partial c} V_1(c,d,n,m,s).$$

Since $S_1(d, n, m, s)$ is the Ring 1 bidders' optimal reduction, matching a reduction of d by all Ring 2 bidders, we have

$$D_1(S_1(d,n,m,s),d,n,m,s) = 0.$$

At $c = S_1(d, n, m, s)$, we must have $c = d$. Hence,

$$D_1(c,c,n,m,s) = 0.$$

$E_1(n, m, s)$ is a solution of this equation for c; as a function of n, m, and s.

From *Equation 1* in ***Lemma 1***, we have

$$\begin{aligned} D_1(c,d,n,m,s) &= \frac{\partial}{\partial c} V_1(c,d,n,m,s) \\ &= \int_{-\infty}^{\infty} (x-c)\cdot F_s^{m-1}(x)\cdot f_s(x)\cdot f_s(x-c+d)\cdot(n-m)\cdot F_s^{n-m-1}(x-c+d)\,dx \\ &\quad + \int_{-\infty}^{\infty} F_s^{m-1}(x)\cdot f_s(x)\cdot F_s^{n-m}(x-c+d)\,dx. \end{aligned}$$

Setting this expression equal to 0 and replacing d with c yields

$$0 = \int_{-\infty}^{\infty} (x-c) \cdot F_s^{n-2}(x) \cdot f_s^2(x) \cdot (n-m)\, dx + \int_{-\infty}^{\infty} F_s^{n-1}(x) \cdot f_s(x)\, dx$$

$$= (n-m) \cdot \int_{-\infty}^{\infty} x \cdot F_s^{n-2}(x) \cdot f_s^2(x)\, dx - c \cdot (n-m) \cdot \int_{-\infty}^{\infty} F_s^{n-2}(x) \cdot f_s^2(x)\, dx + \int_{-\infty}^{\infty} F_s^{n-1}(x) \cdot f_s(x)\, dx$$

$$= (n-m) \cdot \int_{-\infty}^{\infty} x \cdot F_s^{n-2}(x) \cdot f_s^2(x)\, dx - c \cdot (n-m) \cdot \int_{-\infty}^{\infty} F_s^{n-2}(x) \cdot f_s^2(x)\, dx + \frac{1}{n}.$$

Clearly, this has a unique solution for c, and hence for $E_1(n, m, s)$.

$$E_1(n,m,s) = c = \frac{\frac{1}{n} + (n-m) \cdot \int_{-\infty}^{\infty} x \cdot F_s^{n-2}(x) \cdot f_s^2(x)\, dx}{(n-m) \cdot \int_{-\infty}^{\infty} F_s^{n-2}(x) \cdot f_s^2(x)\, dx}$$

$$= \frac{\frac{1}{n \cdot (n-m)} + \int_{-\infty}^{\infty} x \cdot F_s^{n-2}(x) \cdot f_s^2(x)\, dx}{\int_{-\infty}^{\infty} F_s^{n-2}(x) \cdot f_s^2(x)\, dx}$$

This establishes *Equation 2*.

As an example, we find that $E_1(10, 1, 15) \cong 26.438008$, rounded to six decimal places. This is in good agreement with our graphical result.

The Nash equilibrium reduction is directly proportional to s, with $E_1(n, m, s) = s \cdot E_1(n, m, 1)$. It is possible to compute exact values for some error distributions, and certain values of n and m.

***R* normal:** $E_1(2,1,s) = \sqrt{p} \cdot s$ and $E_1(3,1,s) = \frac{\sqrt{3} + 2 \cdot p}{3 \cdot \sqrt{p}} \cdot s$.

***R* uniform on $[-\sqrt{3} \cdot s, \sqrt{3} \cdot s]$:** $E_1(n,m,s) = \frac{\left(n^2 - n \cdot m + 2 \cdot m - 2\right)}{n^2 - n \cdot m} \cdot \sqrt{3} \cdot s$ for all n and m.

The relation between $E_1(n, m, s)$ and reductions in the first and second strategies is interesting, but somewhat unclear. It is obvious that $C_1(n, s) < E_1(n, m, s)$ for all distributions and all values of n, m, and s. The connection of $E_1(n, m, s)$ and $C_1(n, s) + B_1(n, s)$ is more complicated.

R Normal: For $m = 1$ we have

$$C_1(n, s) < E_1(n, 1, s) < C_1(n, s) + B_1(n, s)$$

for all s and $n > 2$. A typical plot is shown in *Figure 4*.

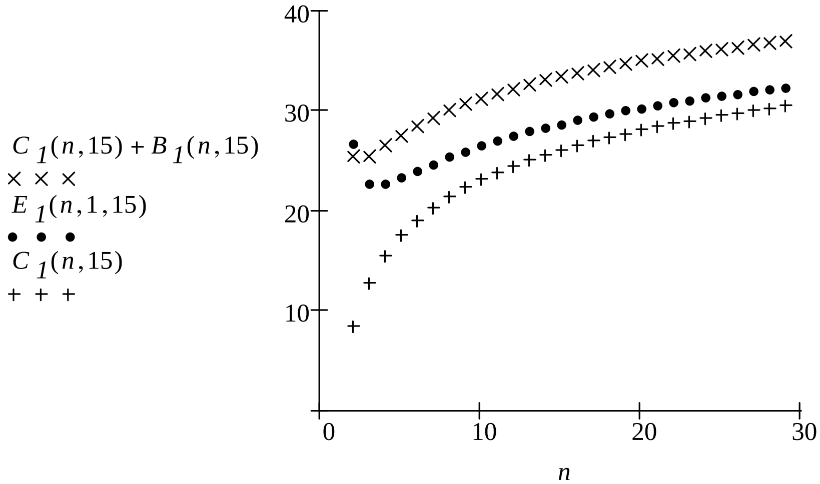

Figure 4. Normal Error Distribution.

For $m > 1$, we have $C_1(n, s) < E_1(n, m, s) < C_1(n, s) + B_1(n, s)$ for all s and all sufficiently large n. A typical plot is shown in *Figure 5*.

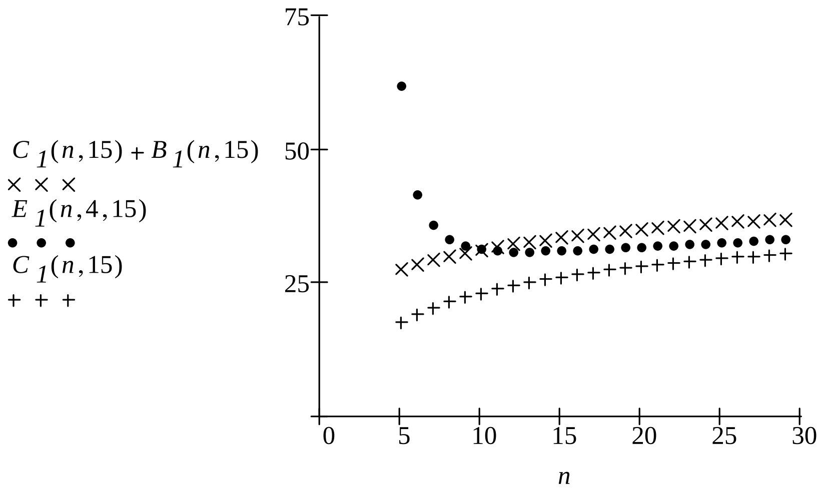

Figure 5. Normal Error Distribution.

There does not seem to be any intuitively obvious explanation for the anomalies at small values of n. In fact, this behavior is a property of the normal distribution, which may or may not occur with other distributions.

R Uniform on $[-\sqrt{3} \cdot s, \sqrt{3} \cdot s]$: In this case, $C_1(n, s) + B_1(n, s) = \sqrt{3} \cdot s$ for all n, and $E_1(n, 1, s) = \sqrt{3} \cdot s$ for all n. Hence, for all n,

$$E_1(n, 1, s) = C_1(n, s) + B_1(n, s).$$

It is somewhat surprising that, for a uniform error distribution, the Nash equilibrium, $E_1(n, 1, s)$ is constant with respect to n. The particular value of $E_1(n, 1, s)$ also has an interesting practical application to auctions. Note that the largest possible signal error is $\sqrt{3} \cdot s$. Hence, a winning bidder that reduced its signal by $E_1(n, 1, s)$ is certain to buy the item at, or below, its actual value.

Continuing with our example of uniform signal errors, we see that for $m > 1$, $\frac{\left(n^2 - n \cdot m + 2 \cdot m - 2\right)}{n^2 - n \cdot m} > 1$, for all n. Hence, for all n,

$$C_1(n, s) + B_1(n, s) < E_1(n, m, s).$$

It appears, but is not yet proven, that, for any error distribution, any m, and any s, $E_1(n, m, s)$ approaches $C_1(n, s)$ as n increases without bound.

Equilibrium Bidding Strategy. We will refer to the signal reduction of all bidders by $E_1(n, m, s)$ as the ***equilibrium bidding strategy***. Note that the winning bidder from either ring has an expected profit of $E_1(n, m, s) - C_1(n, s)$. Since all bidders are equally likely to win the auction, the expected profit for any bidder is $(E_1(n, m, s) - C_1(n, s))/n$.

This is good news for bidders. For normally distributed errors, we find that $E_1(10, 1, 15) - C_1(10, 15) \cong 3.36$, which is approximately 22% of the error standard deviation. The actual economic significance of this can be better appreciated by realizing that the monetary unit in an auction may well be millions of dollars.

From the seller's point of view, the equilibrium bidding strategy is bad news. Confronted with knowledgeable bidders who follow this strategy, the item being auctioned will, on average, sell for below its actual value. Our analysis reveals two interesting aspects of the auction process. First, since the winners curse and the equilibrium value are both proportional to s, the expected difference between the selling price and the actual value is also proportional to s. Hence, better estimation of the item's value by the bidders, resulting in a smaller s, will reduce the seller's expected loss. This establishes a principle that has been found in other auction models. ***The seller should divulge any information that he or she has about the value of the item to be auctioned.*** Such an action will improve the bidders' estimates and reduce the seller's expected loss.

Numerical work indicates that, for fixed m,

$$\lim_{n \to \infty} \left(E_1(n, m, s) - C_1(n, s)\right) = 0,$$

for all s, and any error distribution. This is certainly the case for uniform distributions, where

$$E_1(n, m, s) - C_1(n, s) = \left(\frac{n^2 - n \cdot m + 2 \cdot m - 2}{n^2 - n \cdot m} - \frac{n-1}{n+1}\right) \cdot \sqrt{3} \cdot s$$

$$= 2 \cdot \left(\frac{n^2 - n + m - 1}{n^3 + n^2 \cdot (1-m) - n \cdot m}\right) \cdot \sqrt{3} \cdot s\,.$$

As a consequence, a seller's expected loss will be reduced if he or she can attract more bidders. In our example of normally distributed errors with $m = 1$ and $s = 15$, increasing the number of bidders from 10 to 35 would cut the seller's expected loss in half.

SECOND-PRICE SEALED-BID AUCTIONS

Basic Bidding Strategies. In an attempt to improve the auction mechanism, many sellers have switched to a second-price sealed-bid model. In this case all bidders submit secret bids, with the item being sold to the highest bidder, ***at the price of the second highest bid***. We will consider such auctions in the same setting that we have used for first-price sealed-bid auctions.

The item being sold has an unknown, but common value to each of n bidders, all of whose signals have an error distribution of R. As before, we assume that $m_R = 0$ and use f_s and F_s for the density and distribution functions of R, respectively. It is still possible to consider a partition of the bidders into two bidding rings. However, the complexity of the resulting equations leads us to analyze only the case of $m = 1$ for single bidders. Unless specifically noted to the contrary, all of our results apply to any sufficiently smooth continuous distribution.

The winner's curse and blessing in second-price auctions are defined with order statistics in a manner that is analogous to first-price auctions.

$$C_2(n, s) = L(n, s, n-1) \qquad B_2(n, s) = L(n, s, n-1) - L(n, s, n-2), \text{ for } n > 2$$

As before, for a fixed n, both the curse and blessing are directly proportional to s, with the constants of proportionality depending upon n. The formulas for the curse and blessing have simple forms for specific error distributions and numbers of bidders.

***R* normal:** $C_2(2, s) = -\dfrac{s}{\sqrt{p}}$ and $C_2(3, s) = 0$.

***R* uniform on $[-\sqrt{3} \cdot s, \sqrt{3} \cdot s]$:** $C_2(n, s) = \dfrac{n-3}{n+1} \cdot \sqrt{3} \cdot s$ for all n and

$B_2(n, s) = \dfrac{2}{n+1} \cdot \sqrt{3} \cdot s$ for all $n > 2$.

The ***first bidding strategy***, in a second-price auction, is followed if each bidder submits a bid that is its signal minus $C_2(n, s)$. By analogy with first-price auctions, we let $V_2(c, d, n, s)$ be the expected value of profit for a single bidder that results from its signal reduction of c and a signal reduction of d by all other bidders in a second-price auction. With this strategy, $V_2(C_2(n, s), C_2(n, s), n, s) = 0$ and, on average, the item is sold for its actual value. In the ***second bidding strategy***, in a second-price auction, is followed if each bidder reduces its signal by $C_2(n, s) + B_2(n, s)$.

We now consider the expected profit of a single bidder.

Lemma 2. Suppose that, under our common-value model, in a second-price sealed-bid auction with n bidders, a single bidder bids its signal minus an amount c and the $n-1$ other bidders bid their signals minus an amount d. For any distribution of signal errors having a standard deviation of s, with density f_s, and distribution F_s, we have the following expected value for the single bidder.

$$V_2(c,d,n,s) = -(n-1)\cdot \int_{-\infty}^{\infty} (x-d)\cdot f_s(x)\cdot F_s^{n-2}(x)\cdot [1-F_s(x+c-d)]\,dx \qquad (3)$$

<u>Proof</u>. Let R_1, R_2, …, R_n, M_1, M_2, F_s and f_s be as in the proof of ***Lemma 1***. We now have

$$V_2(c,d,n,m,s) = -\int_{-\infty}^{\infty} (x-d)\cdot\left[\int_{x+c-d}^{\infty} f_s(y)\,dy\right]\cdot(n-1)\cdot f_s(x)\cdot F_s^{n-2}(x)\,dx$$

$$= -(n-1)\cdot \int_{-\infty}^{\infty} (x-d)\cdot f_s(x)\cdot F_s^{n-2}(x)\cdot [1-F_s(x+c-d)]\,dx,$$

which establishes *Equation 3*.

Plots of V_2 are very similar to those of V_1. *Figures 6* and *7* show some typical examples, using a normal error distribution.

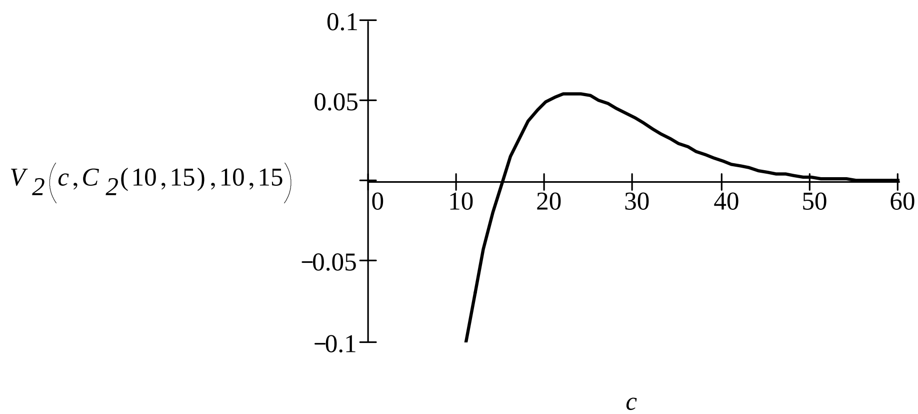

Figure 6. Normal Error Distribution.

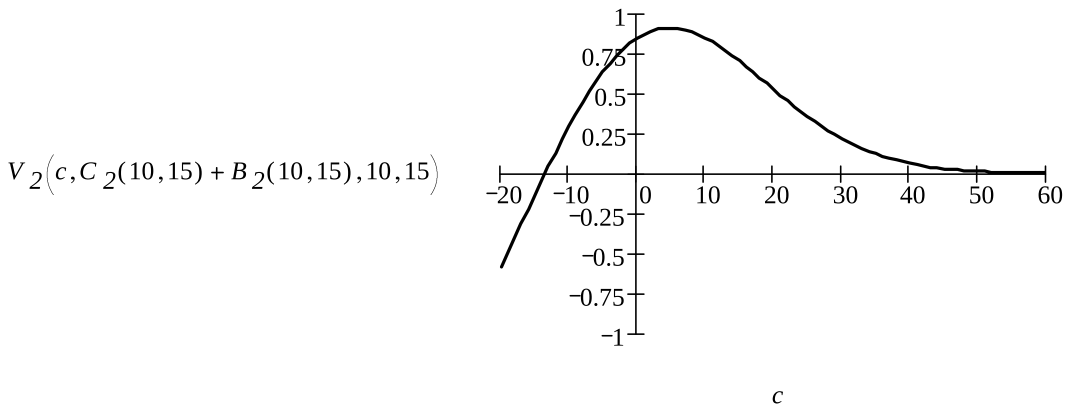

Figure 7. Normal Error Distribution.

It is clear that a single bidder has an optimal response to a fixed signal reduction by all other bidders. *Figures 6* and *7* show that neither the first nor the second bidding strategy is stable. The single bidder should answer a reduction of $C_2(10, 15) \cong 15.0$ with a larger reduction of approximately 22.8. If all others reduce their signals by $C_2(10, 15) + B_2(10, 15) \cong 20.2$, then the single bidder should reduce by the smaller amount of approximately 4.8.

Equilibrium. We, again, have the suggestion of a Nash equilibrium. Let $S_2(d, n, s)$, be the reduction in a single bidder's signal that maximizes its expected value, given that all of the other $n - 1$ bidders reduce their signals by d.

Theorem 2. Consider our common-value model, for a second-price sealed-bid auction with n bidders. For any distribution of signal errors having a standard deviation of s, with density f_s, and distribution F_s, there exists a unique value $E_2(n, s)$ such that $S_2(E_2(n, s), n, s) = E_2(n, s)$. Moreover,

$$E_2(n,s) = \frac{\int_{-\infty}^{\infty} x \cdot F_s^{n-2}(x) \cdot f_s^2(x)\,dx}{\int_{-\infty}^{\infty} F_s^{n-2}(x) \cdot f_s^2(x)\,dx}. \qquad (4)$$

Proof. By analogy with the proof of ***Theorem 1***, let

$$D_2(c,d,n,s) = \frac{\partial}{\partial c} V_2(c,d,n,s),$$

and note that $E_2(n, s)$ must be a solution 0f $D_2(c, c, n, s) = 0$ for c; as a function of n and s.

From *Equation 3* in ***Lemma 2***, we have

$$D_2(c,d,n,s) = \frac{\partial}{\partial c} V_2(c,d,n,s) = (n-1) \cdot \int_{-\infty}^{\infty} (x-d) \cdot f_s(x) \cdot F_s^{n-2}(x) \cdot f_s(x+c-d)\,dx.$$

Setting this expression equal to 0 and replacing d with c yields

$$0 = (n-1) \cdot \int_{-\infty}^{\infty} (x-c) \cdot F_s^{n-2}(x) \cdot f_s^2(x)\,dx$$

$$0 = \int_{-\infty}^{\infty} x \cdot F_s^{n-2}(x) \cdot f_s^2(x)\,dx - c \cdot \int_{-\infty}^{\infty} F_s^{n-2}(x) \cdot f_s^2(x)\,dx.$$

It is clear that

$$E_2(n, s) = c = \frac{\int_{-\infty}^{\infty} x \cdot F_s^{n-2}(x) \cdot f_s^2(x)\,dx}{\int_{-\infty}^{\infty} F_s^{n-2}(x) \cdot f_s^2(x)\,dx}$$

is the unique solution for the Nash equilibrium. This completes the derivation of *Equation 4*.

As is the case with $C_2(n, s)$ and $B_2(n, s)$, the Nash equilibrium reduction, $E_2(n, s)$, is directly proportional to s. It is possible to compute exact values for some error distributions, and certain values of n.

***R* normal:** $E_2(2, s) = 0$ and $E_2(3, s) = \dfrac{s}{\sqrt{3 \cdot p}}$.

***R* uniform on $[-\sqrt{3} \cdot s, \sqrt{3} \cdot s]$:** $E_2(n, s) = \dfrac{n-2}{n} \cdot \sqrt{3} \cdot s$ for all n.

The general relationship between $E_2(n, s)$ and reductions in the first and second strategies is not yet completely established. However, the connections are clear for our sample distributions.

R Normal: For $n = 2$, our formulas show that $C_2(2, s) < E_2(2, s)$. The plot in *Figure 8* indicates that $C_2(n, s) < E_2(n, s) < C_2(n, s) + B_2(n, s)$ for all $n > 2$. Since all of these quantities are proportional to s, the alignment that is shown for $s = 15$ applies to all values of s.

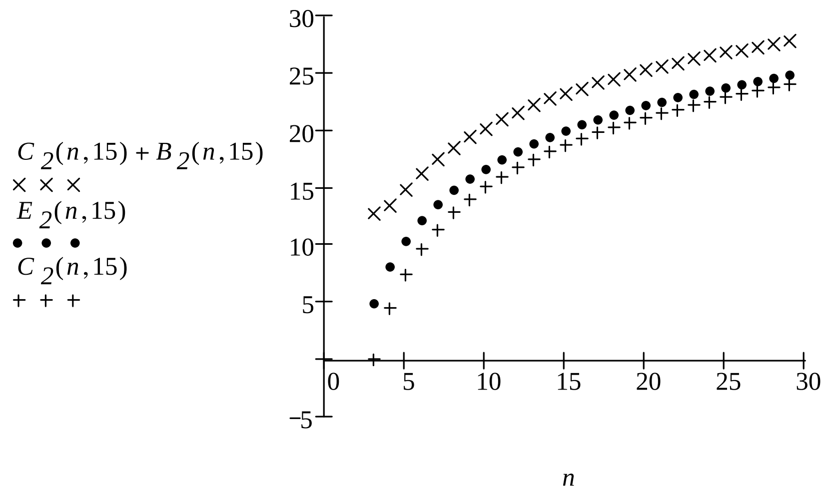

Figure 8. Normal Error Distribution.

R Uniform on $[-\sqrt{3} \cdot s, \sqrt{3} \cdot s]$: Our formulas show that $C_2(2, s) < E_2(2, s)$ and that $C_2(n, s) < E_2(n, s) < C_2(n, s) + B_2(n, s)$ for all $n > 2$.

Equilibrium Bidding Strategy. We will refer to the signal reduction of all bidders by $E_2(n, s)$ as the ***equilibrium bidding strategy*** for second-price auctions. With this strategy, any winning bidder has an expected profit of $E_2(n, s) - C_2(n, s)$. Since all bidders are equally likely to win the auction, the expected profit for any one bidder is $(E_2(n, s) - C_2(n, s))/n$. This is favorable to bidders, but not as profitable as is the case with first-price auctions.

R Normal: We find that $E_2(10, 15) - C_2(10, 15) \cong 1.67$, which is approximately 11% of the error standard deviation. However, it is not nearly as large as was the expected profit for a winning bidder under the equilibrium bidding strategy for first-price auctions. Recall that the corresponding expected profit in that case was $E_1(10, 1, 15) - C_1(10, 15) \cong 3.36$, which is approximately 22% of the error standard deviation. As *Figure 9* shows, the lower expected profit for second-price auctions persists for all numbers of bidders, but is relatively less for larger values of n.

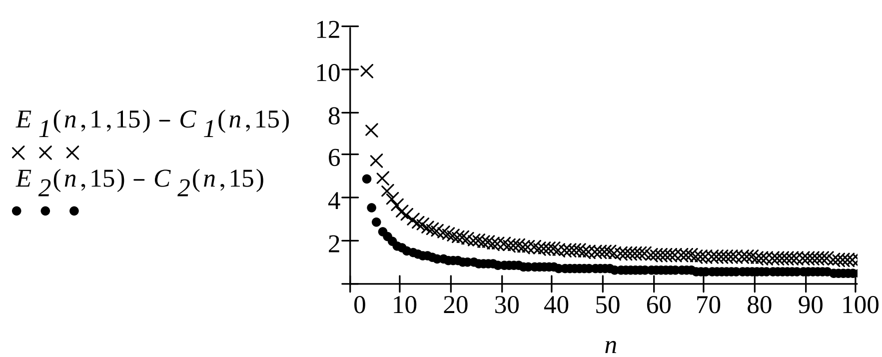

Figure 9. Normal Error Distribution.

Obviously, a seller will fare better with second-price auctions than with first-price sealed-bid auctions. He or she can still expect to sell the item for less than its actual value, but the expected loss will only be approximately half of what it would be under the first-price plan.

R Uniform on $[-\sqrt{3}\cdot s\ ,\ \sqrt{3}\cdot s\ ]$: We find that $E_2(n,s) - C_2(n,s) = \dfrac{2\cdot(n-1)}{n\cdot(n+1)}\cdot\sqrt{3}\cdot s$ and $E_1(n,s) - C_1(n,s) = \dfrac{2\cdot n}{n\cdot(n+1)}\cdot\sqrt{3}\cdot s$. Thus, in the case of only 2 bidders, the expected profit for a winning bidder in second-price auctions is one-half what it is in first-price auctions. As the number of bidders increases both the absolute and the relative advantage to bidders in second-price auctions decreases to zero. As in the normal signal error example, a seller would prefer second-price auctions for uniformly distibuted signal errors.

Our analysis of second-price auctions yields the same information for sellers as we found in the first-price case. The seller should divulge any information that he or she has about the value of the item to be auctioned and should attempt to attract as many bidders as possible.

Revenue Equivalence Theorem. Our results on the expected losses for sellers in first- and second-price auctions are of particular interest in light of the major ***Revenue Equivalence Theorem***. See Klemperer (1999) and McAfee & McMillan (1987). Roughly phrased, this theorem states that, for a wide class of auction models, the expected revenue for the seller is identical under Dutch, English, second-price sealed-bid, and first-price sealed-bid auctions. The Revenue Equivalence Theorem has been shown to apply to private-value models and to some common-value models. ***Our work shows that the Revenue Equivalence Theorem does not hold for our type of common-value model.*** In this case there is a distinct advantage to the seller from choosing a second-price sealed-bid auction method, rather than a first-price sealed-bid sale

Simulation. Many of our results were initially discovered by extensive computer simulation. Simulations for several error distributions, and many values of *n*, *m*, and s also provide numerical verification for our conclusions.

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