Lecture 1

Welcome to Math 129.

In principle, Math 129 is "Calculus II".

Let me first tell you a bit about what I expect you to remember from "Calculus I".

The primary focus of Calculus I is on derivatives.

Given a function $y = f(x)$, one can calculate the rate of change of $f$ at each value of $x$. This creates a new function $y = f'(x)$ which is called the derivative of $f$.

Notation

$$f'(x) = \frac{df}{dx} f(x).$$
A first step in calculus is to understand how to calculate the derivative of several basic functions.

Here is a table.

**Differentiation Table**

<table>
<thead>
<tr>
<th></th>
<th>$f(x)$</th>
<th>$f'(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$x^2$</td>
<td>$2x$</td>
</tr>
<tr>
<td>2</td>
<td>$\sin(x)$</td>
<td>$\cos(x)$</td>
</tr>
<tr>
<td>3</td>
<td>$\tan(x)$</td>
<td>$\sec^2(x)$</td>
</tr>
<tr>
<td>4</td>
<td>$e^x$</td>
<td>$e^x$</td>
</tr>
<tr>
<td>5</td>
<td>$\ln(x)$</td>
<td>$\frac{1}{x}$</td>
</tr>
</tbody>
</table>

I expect you to know these (and more) basic derivatives.

See the worksheets on Math 129 homepage for a complete list.
There are also:

**Differentiation Rules**

1. **The sum rule:**
   
   If \( f \) and \( g \) are two functions, then
   
   \[
   (f + g)'(x) = f'(x) + g'(x)
   \]

2. **The product rule:**

   If \( f \) and \( g \) are two functions, then
   
   \[
   (fg)'(x) = f'(x)g(x) + f(x)g'(x)
   \]

3. **The chain rule:**

   If \( f \) and \( g \) are two functions, then
   
   \[
   (f \circ g)'(x) = f'(g(x)) \cdot g'(x)
   \]

   Recall: \( (f \circ g)(x) = f(g(x)) \) is the composition of \( f \) and \( g \).
In addition to calculating derivatives and learning the rules of differentiation, we also learned how to interpret derivatives in Calculus I.

Given a function \( y = f(x) \):

- if \( f'(x) > 0 \), then \( f \) is increasing at \( x \)
- if \( f'(x) < 0 \), then \( f \) is decreasing at \( x \)
- if \( f'(x) = 0 \), then \( f \) is constant at \( x \)

Towards the end of Calculus I, you also learned about anti-derivatives.

Given a function \( y = f(x) \), can you find another function \( y = F(x) \) for which

\[
F'(x) = f(x).
\]

Any such function \( F \) is called an anti-derivative of \( f \).
The Fundamental Theorem of Calculus tells us that if \( f \) is continuous, then

\[
F(x) = \int f(x) \, dx
\]

is an anti-derivative of \( f \).

In words, anti-derivatives may be calculated by taking indefinite integrals.

Note: If any constant \( C \),

\[
F(x) = \int f(x) \, dx + C
\]

is also an anti-derivative of \( f \).
Integrals also have a geometric interpretation.

Let $y = f(x)$ where $f(x) > 0$.

The shaded region above represents the area under the curve $y = f(x)$ above the $x$-axis and between the lines $x = a$ and $x = b$.

We learned in Calculus I that

\[
\text{Area under } y = f(x) \quad \text{from } x = a \quad \text{to } x = b = \int_{a}^{b} f(x) \, dx = F(b) - F(a)
\]

for any anti-derivative $F$ of $f$. 
Integrals

Two types

Indefinite integrals

\[ \int f(x) \, dx \] is a function (an anti-derivative)

Definite integrals

\[ \int_{a}^{b} f(x) \, dx \] is a number (area)
The main goal of Calculus II is to learn more techniques of integration.

Let us recall some basics:

Every differentiation formula gives us an integration formula:

<table>
<thead>
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<th>Differentiation Table</th>
<th>Integration Table</th>
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<tbody>
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<td>( f(x) )</td>
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See Math 129 homepage for a worksheet on "known" integrals.
In a very similar way, all of our differentiation rules also give us integration rules.

1) **Sum rule:**

If $f$ and $g$ are functions, then

\[
(f + g)'(x) = f'(x) + g'(x).
\]

\[
\Rightarrow \int f'(x) + g'(x)\,dx = \int (f + g)'(x)\,dx = (f + g)(x)
\]

**Fundamental Theorem of Calculus.**

Thus integrals are linear:

\[
\int (f + g)\,dx = \int f\,dx + \int g\,dx
\]
The chain rule gives us our 1st integration method:

**Substitution**

Recall:

The chain rule

Let $f$ and $g$ be functions, then

$$ (f \circ g)'(x) = f'(g(x)) \cdot g'(x) $$

$$ \Rightarrow \quad \int f'(g(x)) \cdot g'(x) dx = \int (f \circ g)'(x) dx $$

Fundamental Theorem of calculus.

The substitution method is based on this observation. If you are integrating a composition, the integral may simplify if you substitute.
Ex 1

\[ \int 2x^2 \sin(3x^3 + 5) \, dx \]

Use substitution.
Look for composition \( \Rightarrow \sin(3x^3 + 5) \)

Relabel the inner function:

\[ u(x) = 3x^3 + 5 \quad \leftrightarrow \quad u = 3x^3 + 5 \]

\[ \frac{du}{dx(x)} = 9x^2 \quad \leftrightarrow \quad du = 9x^2 \, dx \]

\[ \text{Substitute:} \quad \frac{1}{9} \, du = x^2 \, dx \]

\[ \int 2x^2 \sin(3x^3 + 5) \, dx = \int \sin(u) \cdot 2x^2 \, dx \]

\[ = \int \sin(u) \cdot \frac{2}{9} \, du \]

\[ = \frac{2}{9} \left[ -\cos(u) + C \right] \]

\[ = \frac{2}{9} [ -\cos(3x^3 + 5) + C ] \]

This is a complete substitution because no more \( x \)'s appear.

Always rewrite integrals of original variables.

\[ = -\frac{2}{9} \cos(3x^3 + 5) + C \]
Ex 2

\[ \int e^{t^2 + 1} \, dt \]

Use substitution

\[ u = t^2 + 1 \]
\[ du = 2t \, dt \implies t \, dt = \frac{1}{2} \, du \]

\[
\int e^{t^2 + 1} \, dt = \int e^{u} \cdot \frac{1}{2} \, du
\]

\[ = \frac{1}{2} \int e^{u} \, du \]

\[ = \frac{1}{2} \left[ e^{u} + c \right] \]

\[ = \frac{1}{2} \left[ e^{t^2 + 1} + c \right] \]
Ex 3

\[ \int x^3 \sqrt{x^4 + 5} \, dx \]

Use substitution

let \( u = x^4 + 5 \)

\[ du = 4x^3 \, dx \quad \Rightarrow \quad x^3 \, dx = \frac{1}{4} \, du \]

\[ \Rightarrow \int x^3 \sqrt{x^4 + 5} \, dx = \frac{1}{4} \int \sqrt{u} \cdot x^3 \, dx \]

\[ = \frac{1}{4} \int \sqrt{u} \cdot \frac{1}{4} \, du \]

\[ = \frac{1}{4} \int u^{1/2} \, du \]

\[ = \frac{1}{4} \left[ \frac{u^{3/2}}{3/2} \right] + C \]

\[ = \frac{1}{4} \cdot \frac{2}{3} \left( x^4 + 5 \right)^{3/2} + C \]

\[ = \frac{1}{6} (x^4 + 5)^{3/2} + C \]
2nd integration method

Integration by Parts

This is motivated by the product rule.

Product rule

Let \( f \) and \( g \) be functions

\[
(fg)'(x) = f'(x)g(x) + f(x)g'(x)
\]

Let's relate \( u = f \) and \( v = g' \).

\[
(uv)'(x) = u'(x)v(x) + u(x)v'(x).
\]

Integrate both sides:

\[
\int u'(x)v(x) + u(x)v'(x)\,dx = \int (uv)'(x)\,dx
\]

Select for this:

\[
\int (uv)'(x)\,dx = (uv)(x)
\]

Fundamental Theorem.

\[
\int u(x)v'(x)\,dx = (uv)(x) - \int u'(x)v(x)\,dx
\]
Using the product rule we found that

$$\text{Su} \times \text{dv} \, dx = (uv)dx - Su \, dv \, dx$$

**Shorthand**

$$\text{Suv} = uv - Su \, dv$$

This is the famous integration by parts formula.

Much like substitution, this formula allows you to rewrite a complicated integral in a form that may be easier to calculate.
Ex. 1

\[ \int xe^x \, dx \]

Use integration by parts

\[ u = x \quad \Rightarrow \quad du = dx \]
\[ dv = e^x \, dx \quad \Rightarrow \quad v = e^x \]

By IBP formula:

\[ \int uv \, dx = uv - \int vu \, dx \]

\[ = xe^x - \int e^x \cdot x \, dx \]

\[ = xe^x - e^x + C \]
Ex 2

\[ \int x \sin(3x) \, dx \]

**Use integration by parts**

\[ u = x \quad \Rightarrow \quad du = dx \]
\[ dv = \sin(3x) \, dx \quad \Rightarrow \quad v = -\frac{\cos(3x)}{3} \]

\[ \int x \sin(3x) \, dx = uv - \int v \, du \]

\[ = x \left( -\frac{\cos(3x)}{3} \right) - \int \left( -\frac{\cos(3x)}{3} \right) \, dx \]

\[ = -\frac{x \cos(3x)}{3} + \frac{1}{3} \int \cos(3x) \, dx \]

\[ = -\frac{x}{3} \cos(3x) + \frac{1}{3} \cdot \frac{\sin(3x)}{3} + C \]
Integration by Parts

Also works for definite integrals

\[ \int_a^b u \, dv = uv \bigg|_a^b - \int_a^b v \, du \]

Note: \[ uv \bigg|_a^b = uv(b)v(a) - uv(a)v(b) \]

Ex.

\[ \int_0^1 ze^{-z} \, dz \]

Let \( u = z \) \( \Rightarrow \) \( du = dz \)

\( dv = e^{-z} \, dz \) \( \Rightarrow \) \( v = -e^{-z} \)

\[ \int_0^1 ze^{-z} \, dz = z(-e^{-z}) \bigg|_0^1 - \int_0^1 (-e^{-z}) \, dz \]

\[ = -e - 0 + \int_0^1 e^{-z} \, dz \]

\[ = -e - e^2 - e^{-1} - e^{-1} + 1 \]

Answer: \[ 1 - 2e^{-1} \]