Section 11.6 (Word Problems)

1) Water Reservoir

- Initially full with 100 million gallons of water

\[ \text{0.9 million/day (inflow)} \]

\[ \text{0.1 million/day (outflow)} \]

\[ \text{1 million gallons/day to the city} \]

Note: amount in = amount out

\[ \Rightarrow \text{Constantly 100 million gallons of water in reservoir} \]

The question asks for concentration:

- Let \( C(t) \) be the concentration of salt in the reservoir after \( t \) days.

- Let \( Q(t) \) be the amount of salt (in pounds) in the reservoir after \( t \) days.

Note: \( C(t) = \frac{Q(t)}{100 \text{ million gallons}} \) has units of \( \text{lbs of salt/gallon of mixt} \)
It is "easy" to verbalize the D.E. for \( Q \).

\[
\text{Rate of change} = \text{rate of salt in} - \text{rate of salt out} - \text{rate of salt}.
\]

\[
\text{Rate of salt in} = (\text{concentration})(\text{volume of salt in/day})
= (0.0001)(0.1 \text{ million gallons/day})
= 0.0001 \cdot 10^6
= 10 \text{ lbs/day}
\]

\[
\text{Rate of salt out} = (\text{concentration})(\text{volume of salt out no-day})
= \left(\frac{Q(t)}{100 \text{ million gallons}}\right)(1 \text{ million gallons/day})
= \frac{Q(t)}{100} \text{ lbs/day}
\]

\[
\Rightarrow \frac{dQ}{dt} = 10 - \frac{Q}{100}
\]

\[
\Rightarrow \frac{dQ}{dt} = -\frac{1}{100}(Q - 1000) \quad \text{This is a separable D.E.}
\]
The solution (solve like Newton's law problems) is

\[ Q(t) = 1000 + A e^{-\frac{t}{100}} \]

Since

\[ 0 = Q(0) = 1000 + A \implies A = -1000. \]

\[ \implies Q(t) = 1000 - 1000 e^{-\frac{t}{100}} \]

\[ \implies C(t) = \frac{Q(4)}{100 \text{ million gallons}} = \frac{1000 \left(1 - e^{-\frac{4}{100}}\right)}{100 \times 10^6} \text{ gallons} \]

\[ = (10^{-5}) \left(1 - e^{-\frac{1}{25}}\right) \]

2a) \[ \frac{dD}{dt} = -k\sqrt{D} \]

Since the water is leaking out of the barrel, the depth goes down (decrease) as time goes on. Thus, we need \(-k\) in the equation.
b) This is a separable equation:

\[ \frac{dD}{dt} = -k \sqrt{D} \quad D(0) = 25 \text{ in.} \]

**Step 1**: \( \sqrt{D} = 0 \) shows that \( D = 0 \)

is the only constant solution.

Since this is not the solution we want \((D(0) = 25!\)) , we must go on.

**Step 2/3**

\[ \frac{1}{\sqrt{D}} \frac{dD}{dt} = -k \]

\[ \int \frac{1}{\sqrt{D}} dD = - \int k \, dt + C \]

\[ \sqrt{D} = -kt + C \]

\[ \sqrt{D} = -\frac{kt}{2} + \frac{C}{2} \]

\[ \Rightarrow \quad D(t) = \left( -\frac{kt}{2} + \frac{C}{2} \right)^2 \]

Using the initial condition

\[ 25 = D(0) = \left( \frac{C}{2} \right)^2 \quad \Rightarrow \quad 5 = \frac{C}{2} \]

\[ \Rightarrow \quad C = 10 \]
Thus
\[ D(4) = \left( \frac{-k}{2} + 5 \right)^2 \]

c) \[ 24 = D(1) = \left( \frac{-k}{2} + 5 \right)^2 \]

\[ \Rightarrow \frac{-k}{2} + 5 = \sqrt{24} \Rightarrow k = 10 - 2\sqrt{24} \]

We are looking for the time \( t^* \) at which
\[ 0 = D(t^*) = \left( \frac{-kt^*}{2} + 5 \right)^2 \]

\[ \Rightarrow \frac{-kt^*}{2} + 5 = 0 \]

\[ \Rightarrow t^* = \frac{10}{k} = \frac{10}{10 - 2\sqrt{24}} \approx 49.5 \text{ hours} \]

3) Let \( w(t) \) be the company's net worth in millions of dollars with \( t \) in years.

e) Assume \( w(t) \) is only influenced by revenue and payroll. Then

\[ \frac{dw}{dt} = \text{rate revenue earned} - \text{rate payroll obligations are made} \]

\[ = 0.05w - 200 \]

\[ = 0.05 \left( \frac{w - 200}{0.05} \right) = 0.05 (w - 400) \]
b) This is a separable D.E. much like Newton's Law of Cooling.

**STEP 1** \( W - 4000 = 0 \)

Shows that \( W = 4000 \) is the only constant solution.

**STEP 2** If \( W \neq 4000 \),

\[
\frac{1}{W - 4000} \frac{dW}{dt} = 0.05
\]

has solution \( W(t) = 4000 + A e^{0.05t} \)

\[ W(0) = 3000 \implies A = -1000 \]

\[ \implies W(t) = 4000 - 1000 e^{0.05t} \]

\[ W(0) = 5000 \implies A = +1000 \]

\[ \implies W(t) = 4000 + 1000 e^{0.05t} \]