Review of Existence and Uniqueness: Section 1.2

January 17, 2019
Last class we considered a large class of initial value problems with the form:

\[ x' = f(t, x) \quad \text{and} \quad x(t_0) = x_0 \]  

(1)

Our goal was to state a result which tells us when the equation above has one and only one solution. Such results are typically called a results on existence and uniqueness.

Let us denote by

\[ \frac{\partial f}{\partial x}(t, x) \]

the partial derivative of \( f \) with respect to \( x \).
Theorem (Existence and Uniqueness for 1st order i.v.p.)

Let \( f(t, x) \) be a function that is well-defined for \( a < t < b \) and \( c < x < d \). Suppose that:

1. Both \( f(t, x) \) and \( \frac{\partial f}{\partial x}(t, x) \) are continuous in \( t \) and continuous in \( x \) when \( a < t < b \) and \( c < x < d \).

2. The initial condition lies in these intervals, i.e. \( a < t_0 < b \) and \( c < x_0 < d \).

Under these conditions, the initial value problem (1) has a solution on an interval \( \alpha < t < \beta \) which contains \( t_0 \). Moreover, there is no other solution of (1) on this interval.
Example 1: Consider the following initial value problem

\[ x' = tx \quad \text{with} \quad x(0) = \frac{1}{2} \]

Note that

\[ f(t, x) = tx \quad \text{and} \quad \frac{\partial f}{\partial x}(t, x) = t \]

are both continuous in both \( t \) and \( x \) for all real values of \( t \) and \( x \). In this case, the theorem applies. Moreover, the function

\[ x(t) = \frac{1}{2} e^{\frac{t^2}{2}} \]

is a solution and it is defined for all real \( t \).
Example 2: Consider the initial value problem

\[ x' = 3x^{2/3} \quad \text{with} \quad x(0) = 0 \]

Note that

\[ f(t, x) = 3x^{2/3} \quad \text{and} \quad \frac{\partial f}{\partial x}(t, x) = 2x^{-1/3} \]

are both independent of \( t \). Since \( \frac{\partial f}{\partial x} \) is undefined at \( x = 0 \), the theorem does not apply in the case of this particular initial condition.

One readily checks that the functions

\[ x_1(t) = 0 \quad \text{and} \quad x_2(t) = t^3 \]

which are clearly distinct, are both solutions of this IVP.
Although we will not prove this, the following is an interesting fact.

Fact: In the initial value problem (1) has two distinct solutions, then it has infinitely many solutions.

As a result, we conclude that the initial value problem (1) either has: 0, 1, or infinitely many solutions.
Example 3: Consider the initial value problem

\[ x' = 2tx^2 \quad \text{with} \quad x(0) = 1 \]

Note that

\[ f(t, x) = 2tx^2 \quad \text{and} \quad \frac{\partial f}{\partial x}(t, x) = 4tx \]

are both continuous in both \( t \) and \( x \) for all real values of \( t \) and \( x \). In this case, the theorem applies. Moreover, the function

\[ x(t) = \frac{1}{1 - t^2} \]

is a solution. Note further, however, that this solution is only defined for \(-1 < t < 1\). Thus we only get a local solution, even in cases where \( f \) and \( \frac{\partial f}{\partial x} \) are very well behaved!