Key to Test 1

1) a) Let us write f as

$$f(x) = u(x) + iv(x) \quad \text{for all } x \in \mathbb{R}$$

where $u(x) = Re[f(x)]$ and $v(x) = Im[f(x)]$.
Both u and v are real valued functions with a real domain, and so all the calculus we know applies.
The function $h(x) = (f \circ g)(x)$ can be written as

$$h(x) = u(g(x)) + iv(g(x)) \quad \text{for all } x \in \mathbb{R}$$

where $Re[h(x)] = u(g(x))$ and $Im[h(x)] = v(g(x))$.

In class, we learned that:

$$\begin{aligned} h'(x) &= Re[h(x)]' + i\, Im[h(x)]' \\ &= (u(g(x)))' + i\,(v(g(x)))' \\ &= u'(g(x)) \cdot g'(x) + i\, v'(g(x)) \cdot g'(x) \\ &= g'(x)\left(u'(g(x)) + i\, v'(g(x))\right) \\ &= g'(x) \cdot f'(g(x)) \end{aligned}$$

b) Write f as a composition:

Let $g(x) = 2\cos(x)$ and $h(x) = e^{ix}$

Then

$$f(x) = (h \circ g)(x) = e^{2i\cos(x)}$$

$$\begin{aligned} f'(x) = g'(x) \cdot h'(g(x)) &= -2\sin(x) \cdot i\, e^{2i\cos(x)} \\ &= -2i\sin(x)\, e^{2i\cos(x)} \end{aligned}$$

2) Let $a \in \mathbb{C}$. Write $a = \alpha + i\beta$ with $\alpha = Re[a] \in \mathbb{R}$ and $\beta = Im[a] \in \mathbb{R}$.

As we saw in class,

$$e^{a} = e^{\alpha + i\beta} = e^{\alpha} \cdot e^{i\beta} = e^{\alpha}(\cos(\beta) + i\sin(\beta))$$

and so

$$Re[e^{a}] = e^{\alpha}\cos(\beta) = e^{Re[a]}\cos(Im[a])$$

$$Im[e^{a}] = e^{\alpha}\sin(\beta) = e^{Re[a]}\sin(Im[a]).$$

and

$$|e^{a}| = \sqrt{Re[e^{a}]^{2} + Im[e^{a}]^{2}} = \sqrt{e^{2Re[a]}(\cos(Im[a])^{2} + \sin(Im[a])^{2})}$$

$$= e^{Re[a]}.$$

<u>Thus</u> if $z = e^{3-2i}$, then

$Re[z] = e^{3}\cos(2)$ and $Im[z] = e^{3}\sin(-2) = -e^{3}\sin(2)$

and if $w = e^{z}$ then.

$$Re[w] = e^{Re[z]}\cos(Im[z]) = e^{e^{3}\cos(2)} \cdot \cos(e^{3}\sin(2))$$

$$Im[w] = e^{Re[z]}\sin(Im[z]) = e^{e^{3}\cos(2)} \cdot \sin(-e^{3}\sin(2))$$

and

$$|w| = e^{Re[z]} = e^{e^{3}\cos(2)}$$

3)

$$\|f*g\|_1 = \sum_{n=1}^{q} |(f*g)(n)|$$

$$= \sum_{n=1}^{q} \left| \sum_{m=1}^{q} f(n-m)g(m) \right|$$

$$\leq \sum_{n=1}^{q} \sum_{m=1}^{q} |f(n-m)| \cdot |g(m)|$$

$$= \sum_{m=1}^{q} |g(m)| \sum_{n=1}^{q} |f(n-m)|$$

since f is periodic and this is a sum over q consecutive integers, this is just $\sum_{n=1}^{q} |f(n)| = \|f\|_1$

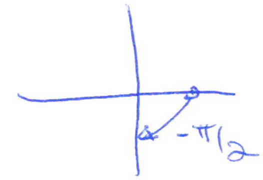
$$= \sum_{m=1}^{q} |g(m)| \cdot \|f\|_1$$

$$= \|f\|_1 \cdot \|g\|_1$$

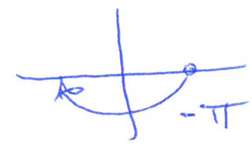
4) i)

$k=0: \quad e\left(-\frac{0}{4}\right) = e(0) = 1.$

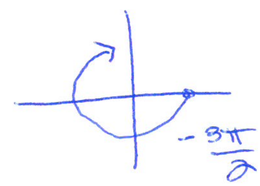
$k=1: \quad e\left(-\frac{1}{4}\right) = e^{2\pi i\left(-\frac{1}{4}\right)} = e^{-\frac{\pi i}{2}} = -i$



$k=2: \quad e\left(-\frac{2}{4}\right) = e^{2\pi i\left(-\frac{1}{2}\right)} = e^{-\pi i} = -1$



$k=3: \quad e\left(-\frac{3}{4}\right) = e^{2\pi i\left(-\frac{3}{4}\right)} = e^{-\frac{3\pi i}{2}} = i$



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ii)

$$\hat{f}(0) = \sum_{n=0}^{3} f(n) = f(0) + f(1) + f(2) + f(3)$$
$$= -1 + a + 1 + (-a)$$
$$= 0.$$

$$\hat{f}(1) = \sum_{n=0}^{3} f(n)\, e\left(-\frac{n}{4}\right) = f(0)e(0) + f(1)\cdot e\left(-\frac{1}{4}\right) + f(2)e\left(-\frac{2}{4}\right) + f(3)e\left(-\frac{3}{4}\right)$$
$$= -1 - ia - 1 - ia$$
$$= -2 - 2ia$$

$$\hat{f}(2) = \sum_{n=0}^{3} f(n)\, e\left(-\frac{2n}{4}\right) = f(0)e(0) + f(1)e\left(-\frac{1}{2}\right) + f(2)e(-1) + f(3)\underbrace{e\left(-\frac{6}{4}\right)}_{= e\left(-\frac{2}{4}\right)}$$
$$= -1 - a + 1 + (-a)(-1)$$
$$= 0$$

$$\hat{f}(3) = \sum_{n=0}^{3} f(n)\, e\left(-\frac{3n}{4}\right) = f(0)e(0) + f(1)e\left(-\frac{3}{4}\right) + f(2)e\left(-\frac{6}{4}\right) + f(3)e\left(-\frac{9}{4}\right)$$
$$= -1 + ia - 1 + ia$$
$$= -2 + 2ia$$