The foundation of analysis is the set of real numbers. We will denote by \( \mathbb{R} \) this set of real numbers, i.e.

\[
\mathbb{R} = \{ x \mid x \text{ is real} \}
\]

For real numbers there are two basic operations: addition and multiplication. A notation for these operations is introduced as follows:

To each pair of real numbers \( x \) and \( y \), there is a unique real number, which we denote by \( x + y \), and refer to as the sum of \( x \) and \( y \). This operation defines addition.

To each pair of real numbers \( x \) and \( y \), there is a unique real number, which we denote by \( xy \), and refer to as the product of \( x \) and \( y \). This operation defines multiplication.

The set of real numbers \( \mathbb{R} \) equipped with these two operations satisfy the field axioms. They are:

**Axiom 1** (Field Axioms). Let \( \mathbb{R} \) be the set of real numbers.

- **Commutativity of Addition**: For any \( x, y \in \mathbb{R} \),

  \[
x + y = y + x.
  \]

- **Associativity of Addition**: For any \( x, y, z \in \mathbb{R} \),

  \[
  (x + y) + z = x + (y + z).
  \]

- **The Additive Identity**: There is a real number, denoted by \( 0 \in \mathbb{R} \), for which

  \[
  0 + x = x + 0 = x \quad \text{for all } x \in \mathbb{R}.
  \]

- **The Additive Inverse**: For each real number \( x \in \mathbb{R} \) there is a real number \( y \in \mathbb{R} \) for which

  \[
  x + y = 0.
  \]

- **Commutativity of Multiplication**: For any \( x, y \in \mathbb{R} \),

  \[
  xy = yx.
  \]

- **Associativity of Multiplication**: For any \( x, y, z \in \mathbb{R} \),

  \[
  (xy)z = x(yz).
  \]
• **The Multiplicative Identity:** There is a real number, denoted by $1 \in \mathbb{R}$, for which

$$1x = x1 = x \quad \text{for all } x \in \mathbb{R}.$$  

• **The Multiplicative Inverse:** For each real number $x \neq 0$, there is a real number $y \in \mathbb{R}$ for which

$$xy = 1.$$  

• **The Distributive Property:** For any $x, y, z \in \mathbb{R}$,

$$x(y + z) = xy + xz.$$  

• **Nontriviality:**

$$1 \neq 0.$$  

**Consequences of the Field Axioms:**

- The additive identity, labeled 0 above, is unique.
- For any $x \in \mathbb{R}$,

$$0x = x0 = 0$$

- For any $x, y \in \mathbb{R}$, if $xy = 0$, then either $x = 0$ or $y = 0$ (both is allowed).
- For any $a \in \mathbb{R}$, there is a unique solution of the equation

$$a + x = 0$$

The solution, which we denote by $x = -a$, is the additive inverse of $a$.

- For any $x, y \in \mathbb{R}$, the difference of $x$ and $y$, which we denote by $x - y$ is defined by

$$x - y = x + (-y)$$

This is how *subtraction* is defined.
- For any $a \in \mathbb{R}$, one has that

$$-(-a) = a$$

- The multiplicative identity, labeled 1 above, is unique.
- For any $a \in \mathbb{R} \setminus \{0\}$, there is a unique solution of the equation

$$ax = 1$$

The solution, which we denote by $x = a^{-1} = \frac{1}{a}$, is the multiplicative inverse of $a$ (also called the reciprocal of $a$).
- For any $x, y \in \mathbb{R}$ with $y \neq 0$, the quotient of $x$ and $y$, which we denote by $x/y$ (or $\frac{x}{y}$) is defined by

$$\frac{x}{y} = xy^{-1}$$
This is how division is defined.

- For any $a \in \mathbb{R} \setminus \{0\}$, one has that
  \[(a^{-1})^{-1} = a\]

- For any $a \in \mathbb{R} \setminus \{0\}$, one has that
  \[(-a)^{-1} = -a^{-1}\]

- By induction, one can prove: Let $a \in \mathbb{R}$ and $n \in \mathbb{N}$. For any $x_1, x_2, \cdots, x_n \in \mathbb{R}$,
  \[a \sum_{k=1}^{n} x_k = \sum_{k=1}^{n} ax_k\]

**Axiom 2 (Positivity Axioms).** There is a subset of $\mathbb{R}$, denoted by $\mathcal{P}$, called the set of positive numbers for which:

- If $x$ and $y$ are positive, then $x + y$ and $xy$ are both positive.
- For each $x \in \mathbb{R}$, exactly one of the following 3 alternatives is true:
  1. $x \in \mathcal{P}$,
  2. $-x \in \mathcal{P}$,
  3. $x = 0$.

**Consequences of the Positivity Axioms:**

- Let $x, y \in \mathbb{R}$. We write $x > y$ if and only if $x - y$ is positive. If $x > y$ we say that $x$ is strictly greater than $y$. We write $x \geq y$ if and only if $x > y$ or $x = y$. If $x \geq y$ we say that $x$ is greater than or equal to $y$.

- Let $x, y \in \mathbb{R}$. We write $x < y$ if and only if $y > x$. If $x < y$ we say $x$ is strictly less than $y$. We write $x \leq y$ if and only if $x < y$ or $x = y$. If $x \leq y$ we say that $x$ is less that or equal to $y$.

- For any $a \in \mathbb{R} \setminus \{0\}$, one has that
  \[a^2 > 0\]
  Since $1 \neq 0$, an application of this to $a = 1$ shows that $1 > 0$.

- For any $a > 0$, one has that
  \[a^{-1} > 0\]

- Let $x, y \in \mathbb{R}$ with $x \leq y$. For any $c \in \mathbb{R}$,
  \[x + c \leq y + c\]
  If the assumed inequality is strict, the resulting inequality is strict as well.
• Let $x_1, x_2, y_1, y_2 \in \mathbb{R}$ with $x_1 \leq y_1$ and $x_2 \leq y_2$. Then,

$$x_1 + x_2 \leq y_1 + y_2.$$ 

If either of the assumed inequalities is strict, then the resulting inequality is strict as well.

An immediate consequence of the above is the following.

• Let $x_1, x_2, y_1, y_2 \in \mathbb{R}$ with $x_1 \leq y_1$ and $x_2 \geq y_2$. Then,

$$x_1 - x_2 \leq y_1 - y_2.$$ 

If either of the assumed inequalities is strict, then the resulting inequality is strict as well.

• If $x > y$, then

$$xc > yc \quad \text{if } c > 0$$ 

and

$$xc < yc \quad \text{if } c < 0$$

• By induction, one can prove: For any $n \in \mathbb{N}$, let $x_1, x_2, \cdots, x_n$ be non-negative numbers, i.e. $x_k \geq 0$ for all $k \in \{1, \cdots, n\}$.

One has that the sum of non-negative numbers is non-negative, i.e.

$$\sum_{k=1}^{n} x_k \geq 0$$

and moreover,

$$\sum_{k=1}^{n} x_k = 0 \quad \text{if and only if} \quad x_1 = x_2 = \cdots = x_n = 0.$$

One has that the product of non-negative numbers is non-negative, i.e.

$$x_1 x_2 \cdots x_n \geq 0$$

and moreover,

$$x_1 x_2 \cdots x_n = 0 \quad \text{if and only if} \quad \text{there is some } k \in \{1, \cdots, n\} \text{ for which } x_k = 0.$$

Chains of Inequalities:

It is sometimes useful to make statements involving multiple inequalities. A valid chain of inequalities (with two links) is a statement of the form:

Let $x, y, z \in \mathbb{R}$.

We write

$$x \leq y \leq z \quad \text{if and only if} \quad x \leq y \quad \text{and} \quad y \leq z$$

In the case above, one checks that $x \leq z$.

We write

$$x \leq y < z \quad \text{if and only if} \quad x \leq y \quad \text{and} \quad y < z$$

In the case above, one checks that $x < z$. 
We write
\[ x < y \leq z \quad \text{if and only if} \quad x < y \quad \text{and} \quad y \leq z \]

In the case above, one checks that \( x < z \).

More valid chains of inequalities are:
\[ x \geq y \geq z, \quad x \geq y > z, \quad \text{and} \quad x > y \geq z. \]

They are defined and have consequences similar to the above statements. These are the only valid chains of inequalities with two links. No other combination has a logical interpretation.

One can extend this notion to chains of inequalities with more than two links. The only valid chains are those for which:
\begin{itemize}
  \item all linking inequalities are either \( \geq \) or \( > \).
  \item all linking inequalities are either \( \leq \) or \( < \).
\end{itemize}

No other combinations have a logical interpretation.