Key To Exam 1 Make Up

1. We start with the following observation.

Let $a, b \in \mathbb{R}$ with $a \geq 0$ and $b \geq 0$.

Case 1: $a = 0$ and $b = 0$. Then $a + b = 0 + 0 = 0$.

Case 2: $a = 0$ and $b > 0$. Then $a + b = 0 + b = b > 0$.

Case 3: $a > 0$ and $b = 0$. Then $a + b = a + 0 = a > 0$.

Case 4: $a > 0$ and $b > 0$. Then $a + b > 0$ by positivity axioms.

Thus we conclude: If $a, b \in \mathbb{R}$ and $a \geq 0$ and $b \geq 0$, then $a + b \geq 0$ and $a + b = 0$ if and only if $a = b = 0$.

We proceed by induction.

Take $S(n) = \text{let } n \in \mathbb{N} \text{ and } x_j \geq 0 \text{ for } 1 \leq j \leq n$, then $\sum_{j=1}^{n} x_j \geq 0$.

Base Case: $S(1)$. Let $x_1 \geq 0$. Then $\sum_{j=1}^{1} x_j = x_1 \geq 0$.

Inductive Step: Suppose $S(n)$ is true.

Let $x_j \geq 0$ for $1 \leq j \leq n+1$.

Take $a = x_{n+1}$ and $b = \sum_{j=1}^{n} x_j$.

By assumption $a = x_{n+1} \geq 0$. By $S(n)$, $b = \sum_{j=1}^{n} x_j \geq 0$.

We conclude $\sum_{j=1}^{n+1} x_j = a + b \geq 0$ by the above.
Take $S(n) = \text{let } n \in \mathbb{N} \text{ and } x_j \geq 0 \text{ for } 1 \leq j \leq n$.

Then
\[
\sum_{j=1}^{n} x_j = 0 \text{ if and only if } x_j = 0 \text{ for } 1 \leq j \leq n.
\]

Base case: $S(1)$. Let $x_1 \geq 0$. Then
\[
\sum_{j=1}^{1} x_j = x_1.
\]

In this case, $\sum_{j=1}^{1} x_j = 0$ if and only if $x_1 = 0$.

Inductive step: Suppose $S(n)$ is true.

Let $x_j \geq 0$ for $1 \leq j \leq n+1$.

Take $a = x_{n+1}$ and $b = \sum_{j=1}^{n} x_j$.

Then
\[
\sum_{j=1}^{n+1} x_j = a + b. \text{ Thus } \sum_{j=1}^{n+1} x_j = 0 \text{ if and only if } a + b = 0.
\]

By above, $a + b = 0$ if and only if $a = 0$ and $b = 0$.

By $S(n)$, $b = 0$ if and only if $x_j = 0$ for $1 \leq j \leq n$.

This completes the argument.
2) a) For each \( n \in \mathbb{N} \), let

\[
\alpha_n = \frac{1}{\sqrt{n+3}} + \frac{n-3n^2}{2n^2-1}
\]

Claim: \( \lim_{n\to\infty} \alpha_n \) converges to \( a = -\frac{3}{2} \).

To prove this, let \( \varepsilon > 0 \). Consider

\[
|\alpha_n - a| = \left| \frac{1}{\sqrt{n+3}} + \frac{n-3n^2}{2n^2-1} + \frac{3}{a} \left( \frac{n^2-\frac{1}{2}}{n^2+\frac{1}{2}} \right) \right|
\]

\[
= \left| \frac{1}{\sqrt{n+3}} + \frac{n-\frac{3}{2}}{2n^2-1} \right|
\]

\[
\leq \frac{1}{\sqrt{n+3}} + \frac{n-\frac{3}{2}}{2n^2-1}
\]

We show there is \( N_1 \in \mathbb{N} \) for which

\[
\frac{1}{\sqrt{n+3}} \leq \frac{\varepsilon}{2} \quad \text{for all } n \geq N_1
\]

and \( N_2 \in \mathbb{N} \) for which

\[
\frac{n-\frac{3}{2}}{2n^2-1} \leq \frac{\varepsilon}{2} \quad \text{for all } n \geq N_2
\]

The claim is proven for \( N = \max\{N_1, N_2\} \), as we conclude that for all \( n \geq N \)

\[
|\alpha_n - a| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]
Let $\varepsilon > 0$ be small enough so that

$$\varepsilon = \left(\frac{a}{\varepsilon}\right)^2 - 3 > 0.$$ 

By the Archimedean Property, there is $N_1 \in \mathbb{N}$ for which

$$\frac{a}{\varepsilon} < N_1 \text{, i.e. } \left(\frac{2}{\varepsilon}\right)^2 - 3 < N_1 \implies \left(\frac{a}{\varepsilon}\right)^2 < N_1 + 3 = \sqrt{N_1 + 3},$$

$$\implies \frac{1}{\sqrt{N_1 + 3}} < \frac{\varepsilon}{a}.$$  

Note that:

$$n - \frac{3}{2} < n - 1 \quad \text{and} \quad 2n^2 - 2 < 2n^2 - 1$$

$$\frac{n - \frac{3}{2}}{2n^2 - 1} \leq \frac{n - 1}{2n^2 - 2} = \frac{n - 1}{2(n + 1)(n - 1)} = \frac{1}{2} \cdot \frac{1}{n + 1}$$

Let $\varepsilon > 0$ be small enough so that

$$\tilde{\varepsilon} = \frac{1}{\varepsilon} - 1 > 0.$$ 

By the Archimedean Property, there is $N_2 \in \mathbb{N}$ for which

$$\frac{a}{\tilde{\varepsilon}} < N_2 \implies \frac{a}{\varepsilon} - 1 < N_1 \implies \frac{1}{\varepsilon} < N_1 + 1 \implies \frac{1}{N_1 + 1} < \frac{1}{\varepsilon} < \frac{\varepsilon}{a}.$$
Consider

\[ x_n = \sqrt{\frac{4n^2 - 1}{n^2 + 1}} \text{ for } n \in \mathbb{N}. \]

We show that \( x_n \) is a bounded, monotone increasing sequence. As such, it has a limit by the Monotone Convergence Theorem (Theorem 2.25).

**Note 1:** By definition \( x_n \geq 0 \). Thus we need only show \( x_n \) is bounded above to conclude that \( x_n \) is bounded.

**Note 2:**

\[ x_n^2 = \frac{4n^2 - 1}{n^2 + 1} = \frac{4(n^2 + 1) - 1}{n^2 + 1} = 4 - \frac{5}{n^2 + 1}, \]

we conclude: \( x_n^2 \leq 4 \), i.e. \( x_n \leq 2 \). (Use H.W. #9, Section 1.3)

Thus \( x_n \) is a bounded sequence.

**Note 3:**

\[ x_{n+1} - x_n = \left( 4 - \frac{5}{(n+1)^2 + 1} \right) - \left( 4 - \frac{5}{n^2 + 1} \right) = \left( \frac{1}{n^2 + 1} - \frac{1}{(n+1)^2 + 1} \right) \]

\[ = 5 \cdot \frac{2n+1}{(n+1)(n+2)(n+3)}, \quad \geq 0. \]
Thus \( x_n^2 \leq x_{n+1} \) \( \Rightarrow \) \( x_n \leq x_{n+1} \) (by H.W. #9, Sect. 1.3)

Thus \( \{x_n\} \) is monotone increasing.

By the MCT, we conclude

\[ x = \lim_{n \to \infty} x_n \text{ exists.} \]

By the product rule for limits,

\[ x^2 = \lim_{n \to \infty} x_n^2 = \lim_{n \to \infty} \frac{4n^2 - 1}{n^2 + 1} = 4. \]

\[ \Rightarrow \quad x = 2. \quad \text{Since } x_n \geq 0 \text{ for all } n \in \mathbb{N}. \]

\[ x = 2 \text{ using Lemma 2.21.} \]

3) Case 1: Suppose \( f(x_n^3) \) is unbounded from above, i.e., for every \( c > 0 \) \( \exists \epsilon > 0 \) for which

\[ f(x_n^3) > c. \]

We use this fact to produce a subsequence.

Take \( c_1 = 1 \). Then there is \( n_1 \in \mathbb{N} \) for which

\[ f(x_{n_1}) > 1. \]
Take $c = 2$. Then there is $n_2 \in \mathbb{N}$ with $n_2 > n_1$, for which
\[ f(x_{n_2}) > 2. \]

Now, suppose $n_k \in \mathbb{N}$ has been chosen with $n_k > n_{k+1}$ and
\[ f(x_{n_k}) > k. \]

Choose $n_{k+1} > n_k$ for which
\[ f(x_{n_{k+1}}) > k+1. \]

This process produces a sequence $\{f(x_{n_k})\}$ which is a subsequence of $\{f(x_n)\}$. Clearly $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$.

Now the sequence $\{x_{n_k}\}$ is in $(0,1] \subset (0,1]$ and by sequential compactness (i.e. Theorem 2.36) there is a subsequence $\{x_{n_k'}\}$ which converges to a point in $(0,1]$.

Claim: This subsequence converges to $0$.

Suppose not. Then there is $x_0 \in (0,1]$ and $\{x_{n_k'}\}$ converges to $x_0$. Since $f$ is continuous
\[ f(x_0) = \lim_{j \to \infty} f(x_{n_k'}). \]
Since \( f(x_0) \in \mathbb{R} \), the Archimedean Property guarantees that there is \( N \in \mathbb{N} \) with

\[ f(x_0) < N. \]

For any \( j \geq N \), \( x_j \geq y_j \)

\[ \Rightarrow f(x_{nj}) = f(y_{nj}) \leq y_j \leq y_{nj} \leq N. \]

Thus \( \{f(x_{nj})\} \) cannot converge to \( x_0 \).

We conclude \( x_0 = 0 \).

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4) Let \( n \in \mathbb{N} \). Suppose

\[ D = \bigcup_{k=1}^{n} \text{Lar, } b_k. \]

a) Show that \( D \) is closed and bounded.

**Bounded**: Take \( A = \min \{a_k \mid 1 \leq k \leq n\} \),

\( B = \max \{b_k \mid 1 \leq k \leq n\} \).

Let \( x \in D \). Then \( x \in \text{Lar, } b_k \) for some \( k \in \{1, 2, \ldots, n\} \).

Thus

\[ A \leq a_k \leq x \leq b_k \leq B \]

and hence \( D \) is bounded (i.e., bounded above and below).
Closed: Let \( \{x_j\} \) be a sequence in \( \mathbb{D} \) which converges to \( x_0 \). 

Claim. There is \( k_0 \in \mathbb{N} \), \( j \), and a subsequence \( \{x_{j_k}\} \) of \( \{x_j\} \) such that \( x_{j_k} \in \mathbb{D} \) and \( x_{j_k} \to x_0 \). 

Suppose not. Then for every \( k \in \mathbb{N} \), there is \( N_k \in \mathbb{N} \) and \( x_j \in \mathbb{R}/\mathbb{Z} \) for all \( j \geq N_k \). 

Take \( N = \max N_k \) for \( 1 \leq k \leq n \). 

Then \( x_{N+1} \in \mathbb{D} \) but \( x_{N+i} \in \mathbb{R}/\mathbb{Z} \) for all \( i \in \mathbb{N} \). 

Therefore, \( x_{N+i} \) is a subsequence of \( \{x_j\} \) that converges to \( x_0 \) as well (Prop. 2.30). 

Since \( \mathbb{D} \) is closed, \( x_0 \in \mathbb{D} \) and \( x_{N+i} \to x_0 \). 

Thus, \( x_0 \in \mathbb{D} \) as claimed.

b) Let \( \{x_j\} \) be a sequence in \( \mathbb{D} \). 

By the claim above, there is a subsequence \( \{x_{j_k}\} \) in \( \mathbb{D} \) for some \( k \in \mathbb{N} \). 

Since \( \mathbb{D} \) is sequentially compact (Theorem 2.37), there is a subsequence of \( \{x_{j_k}\} \) converging to an element of \( \mathbb{D} \). This element is in \( \mathbb{D} \).
c). Without loss of generality, assume all intervals are disjoint. (i.e., \( [a_k, b_k] \cap [a_j, b_j] = \emptyset \) whenever \( k \neq j \).)

Otherwise, there are \( n - 1 \) or fewer closed bounded intervals in \( \mathbb{D} \).

Without loss of generality, assume \( q_1 < q_2 < \ldots < q_n \). (Otherwise, relabel.)

Take \( \delta = \min \{ q_{k+1} - b_k \mid 1 \leq k \leq n - 1 \} \).

Since all intervals are disjoint, \( \delta > 0 \).

Now: for each \( 1 \leq k \leq n \), consider the function \( f_k : [a_k, b_k] \to \mathbb{R} \) with \( f_k(x) = f(x) \) for \( x \neq q_k \).
This \( f_k \) is continuous and by Theorem 3.17, \( f_k \) is uniformly continuous.
Hence, for any \( \varepsilon > 0 \), \( \exists \delta_k > 0 \) such that
\[
|f(x) - f(y)| = \left| f_k(x) - f_k(y) \right| < \varepsilon
\]
whenever \( |x - y| < \delta_k \) for \( x, y \in [a_k, b_k] \).

Claim: Take \( \delta' = \min \{ \delta, \delta_1, \ldots, \delta_n \} \).
For $\delta > 0$ and
\[(f(x) - f(y)) < \varepsilon \text{ if } |x-y| < \delta, \quad \forall x, y \in A.
\]
This is because if $x, y \in A$ and $|x-y| < \delta$, this means $x, y \in \mathbb{C}_{a, k}, \mathbb{C}_{b, k}$ for some $k \in \mathbb{N}$.

\[d) \quad \text{As before, for each } k \in \mathbb{N}, \text{ let } f_k : \mathbb{C}_{a, k}, \mathbb{C}_{b, k} \rightarrow \mathbb{R} \text{ be given by}
\[f_k(x) = f(x) \quad \forall x \in \mathbb{C}_{a, k}, \mathbb{C}_{b, k}.
\]

By Theorem 3.9, the extreme value theorem holds for each $f_k$, thus there is $x_k^{\min}, x_k^{\max} \in \mathbb{C}_{a, k}, \mathbb{C}_{b, k}$ and
\[f_k(x_k^{\min}) \leq f_k(x) \leq f_k(x_k^{\max}) \quad \forall x \in [a, b].
\]
Take $f^- = \min \{ f_k(x_k^{\min}) \mid k \in \mathbb{N} \}$ and $f^+ = \max \{ f_k(x_k^{\max}) \mid k \in \mathbb{N} \}$.

For any $x \in \mathbb{D}$, $x \in \mathbb{C}_{a, k}, \mathbb{C}_{b, k}$ for some $k$ and hence,
\[f^- \leq f_k(x) = f(x) = f(x) \leq f^+.
\]