Key To Exam 1 Make UP

1). We start with the following observation.

Let $a, b \in \mathbb{R}$ with $a \geq 0$ and $b \geq 0$.

Case 1: $a = 0$ and $b = 0$. Then $a+b = 0+0 = 0$.

Case 2: $a = 0$ and $b > 0$, Then $a+b = 0+b = b > 0$.

Case 3: $a > 0$ and $b = 0$. Then $a+b = a+0 = a > 0$.

Case 4: $a > 0$ and $b > 0$. Then $a+b > 0$ by positivity axioms.

Thus we conclude: If $a, b \in \mathbb{R}$ and $a \geq 0$ and $b \geq 0$, then

$a+b \geq 0$ and $a+b = 0$ if and only if $a = b = 0$.

We proceed by induction.

Take $S(n)$ = let $n \in \mathbb{N}$ and $x_j \geq 0$ for $1 \leq j \leq n$, then

$$\sum_{j=1}^{n} x_j \geq 0.$$

Base Case: $S(1)$. Let $x_1 \geq 0$. Then $\sum_{j=1}^{1} x_j = x_1 \geq 0$ ✓

Inductive Step: Suppose $S(n)$ is true.

Let $x_j \geq 0$ for $1 \leq j \leq n+1$.

Take $a = x_{n+1}$ and $b = \sum_{j=1}^{n} x_j$.

By assumption $a = x_{n+1} \geq 0$. By $S(n)$, $b = \sum_{j=1}^{n} x_n \geq 0$.

We conclude

$$\sum_{j=1}^{n+1} x_j = a+b \geq 0 \quad \text{by the above.}$$

Take $S'(n) =$ Let $n \in \mathbb{N}$ and $x_j \geq 0$ for $1 \leq j \leq n$.
Then
$$\sum_{j=1}^{n} x_j = 0 \text{ if and only if } x_j = 0 \text{ for } 1 \leq j \leq n.$$

<u>Base Case</u>: $S'(1)$. Let $x_1 \geq 0$. Then $\sum_{j=1}^{1} x_j = x_1$

In this case $\sum_{j=1}^{1} x_j = 0$ if and only if $x_1 = 0$.

<u>Inductive Step</u> Suppose $S'(n)$ is true.
Let $x_j \geq 0$ for $1 \leq j \leq n+1$.
Take $a = x_{n+1}$ and $b = \sum_{j=1}^{n} x_j$.
<u>Then</u>
$\sum_{j=1}^{n+1} x_j = a + b$. Thus $\sum_{j=1}^{n+1} x_j = 0$ if and only if $a+b=0$

By above, $a+b=0$ if and only if $a=0$ and $b=0$.

By $S(n)$, $b=0$ if and only if $x_j = 0$ for $1 \leq j \leq n$.

This completes the argument.

2) a) For each $n \in \mathbb{N}$, let

$$a_n = \frac{1}{\sqrt{n+3}} + \frac{n-3n^2}{2n^2-1}$$

Claim: $\{a_n\}$ converges to $a = -\frac{3}{2}$.
To prove this, let $\varepsilon > 0$. Consider

$$|a_n - a| = \left| \frac{1}{\sqrt{n+3}} + \frac{n-3n^2}{2n^2-1} + \frac{3}{2}\frac{(n^2-\frac{1}{2})}{(n^2-\frac{1}{2})} \right|$$

$$= \left| \frac{1}{\sqrt{n+3}} + \frac{n-\frac{3}{2}}{2n^2-1} \right|$$

$$\leq \frac{1}{\sqrt{n+3}} + \frac{n-\frac{3}{2}}{2n^2-1}$$

We show there is $N_1 \in \mathbb{N}$ for which

$$\frac{1}{\sqrt{n+3}} \leq \frac{\varepsilon}{2} \quad \text{for all } n \geq N_1$$

and $N_2 \in \mathbb{N}$ for which

$$\frac{n-\frac{3}{2}}{2n^2-1} \leq \frac{\varepsilon}{2} \quad \text{for all } n \geq N_2.$$

The claim is proven for $N = \max\{N_1, N_2\}$, as we conclude that for all $n \geq N$

$$|a_n - a| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Let $\varepsilon > 0$ be small enough so that

$$\tilde{\varepsilon} = \left(\frac{2}{\varepsilon}\right)^2 - 3 > 0.$$

By the Archimedean Property, there is $N_1 \in \mathbb{N}$ for which

$$\tilde{\varepsilon} < N_1 \quad \text{i.e.} \quad \left(\frac{2}{\varepsilon}\right)^2 - 3 < N_1 \implies \left(\frac{2}{\varepsilon}\right)^2 < N_1 + 3$$

Use H.W. #9 from Section 1.3

$$\implies \frac{2}{\varepsilon} < \sqrt{N_1 + 3}$$

$$\implies \frac{1}{\sqrt{N_1 + 3}} < \frac{\varepsilon}{2}. \checkmark$$

<u>Note that:</u>

$$n - 3/2 < n - 1 \quad \text{and} \quad 2n^2 - 2 < 2n^2 - 1$$

$$\implies \frac{n - \frac{3}{2}}{2n^2 - 1} \leq \frac{n-1}{2n^2 - 2} = \frac{n-1}{2(n+1)(n-1)} = \frac{1}{2} \cdot \frac{1}{n+1}$$

Let $\varepsilon > 0$ be small enough so that

$$\tilde{\varepsilon} = \frac{1}{\varepsilon} - 1 > 0.$$

By the Archimedean Property, there is $N_2 \in \mathbb{N}$ for which

$$\tilde{\varepsilon} < N_1 \quad \text{i.e.} \quad \frac{1}{\varepsilon} - 1 < N_1 \implies \frac{1}{\varepsilon} < N_1 + 1 \checkmark$$

$$\implies \frac{1}{N_1 + 1} < \varepsilon \implies \frac{1}{2(N_1 + 1)} < \frac{\varepsilon}{2}.$$

2b) Consider

$$x_n = \sqrt{\frac{4n^2-1}{n^2+1}} \quad \text{for } n \in \mathbb{N}.$$

We show that $\{x_n\}$ is a bounded, monotone increasing sequence. As such, it has a limit by the Monotone Convergence Theorem. (Theorem 2.25).

Note 1: By definition $x_n \geq 0$. Thus we need only show $\{x_n\}$ is bounded above to conclude that $\{x_n\}$ is bounded.

Note 2:

$$x_n^2 = \frac{4n^2-1}{n^2+1} = \frac{4(n^2+1-1)-1}{n^2+1} = 4 - \frac{5}{n^2+1}$$

We conclude: $x_n^2 \leq 4$ i.e. $x_n \leq 2$. (use H.W. #9 Section 1.3)

Thus $\{x_n\}$ is a bounded sequence.

Note 3:

$$x_{n+1}^2 - x_n^2 = \left(4 - \frac{5}{(n+1)^2+1}\right) - \left(4 - \frac{5}{n^2+1}\right)$$

$$= 5 \cdot \left[\frac{1}{n^2+1} - \frac{1}{(n+1)^2+1}\right]$$

$$= 5 \cdot \frac{2n+1}{(n^2+1)((n+1)^2+1)} \geq 0.$$

Thus $x_n^2 \le x_{n+1}^2 \iff x_n \le x_{n+1}$ (by H.W. #9 Section 1.3)

Thus $\{x_n\}$ is monotone increasing.

By the MCT, we conclude

$$x = \lim_{n\to\infty} x_n \text{ exists.}$$

By the Product rule for limits,

$$x^2 = \lim_{n\to\infty} x_n^2 = \lim_{n\to\infty} \frac{4n^2-1}{n^2+1} = 4.$$

$\Rightarrow x = \pm 2$. Since $x_n \ge 0$ for all $n \in \mathbb{N}$.

$x = 2$ using Lemma 2.21.

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3) Case 1. Suppose $\{f(x_n)\}$ is unbounded from above, i.e. for every $c > 0$ $\exists N \in \mathbb{N}$ for which

$$f(x_N) > c.$$

We use this fact to produce a subsequence. Take $c_1 = 1$. Then there is $n_1 \in \mathbb{N}$ for which

$$f(x_{n_1}) > 1.$$

Take $C_2 = 2$. Then there is $n_2 \in \mathbb{N}$ with $n_2 > n_1$ for which
$$f(x_{n_2}) > 2.$$

Now, suppose $n_k \in \mathbb{N}$ has been chosen with $n_k > n_{k+1}$ and
$$f(x_{n_k}) > k.$$

Choose $n_{k+1} > n_k$ for which
$$f(x_{n_{k+1}}) > k+1.$$

This process produces a sequence $\{f(x_{n_k})\}$ which is a subsequence of $\{f(x_n)\}$. Clearly $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$.

Now the sequence $\{x_{n_k}\} \subset (0,1] \subset [0,1]$ and so by sequential compactness (i.e. Theorem 2.36) there is a subsequence $\{x_{n_{k_j}}\}$ which converges to a point in $[0,1]$.

Claim: This subsequence converges to 0.

Suppose not. Then there is $x_0 \in (0,1]$ and $\{x_{n_{k_j}}\}$ converges to x_0. Since f is continuous
$$f(x_0) = \lim_{j \to \infty} f(x_{n_{k_j}}).$$

Since $f(x_0) \in \mathbb{R}$, the Archimedean Property guarantees that there is $N \in \mathbb{N}$ with

$$f(x_0) < N.$$

For any $j \geq N$, $n_{k_j} \geq n_j$

$$\Rightarrow \quad f(x_{n_{k_j}}) \geq n_{k_j} \geq n_j \geq N$$

Thus $\{f(x_{n_{k_j}})\}$ cannot converge to x_0 ↯.

We conclude $x_0 = 0$.

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4) Let $n \in \mathbb{N}$. Suppose

$$\mathcal{D} = \bigcup_{k=1}^{n} [a_k, b_k].$$

a) Show that $\mathcal{D}$ is closed and bounded.

<u>Bounded</u>: Take $A = \min\{a_k \mid 1 \leq k \leq n\}$.

Take $B = \max\{b_k \mid 1 \leq k \leq n\}$.

Let $x \in \mathcal{D}$. Then $x \in [a_{k_0}, b_{k_0}]$ for some $k_0 \in \{1, 2, \ldots$

Thus

$$A \leq a_{k_0} \leq x \leq b_{k_0} \leq B$$

and hence $\mathcal{D}$ is bounded (i.e. bounded above and below)

Closed: Let $\{x_j\}$ be a sequence in $\mathcal{D}$ which converges to x_0.

Claim: There is $k_0 \in \{1, \dots, n\}$ and a subsequence $\{x_{j_k}\}$ of $\{x_j\}$ with $\{x_{j_k}\}$ in $[a_{k_0}, b_{k_0}]$.

Suppose not. Then for every $1 \le k \le n$ there is $N_k \in \mathbb{N}$ and $x_j \in \mathbb{R}/[a_k, b_k]$ for all $j \ge N_k$.

Take $N = \max\{N_k \mid 1 \le k \le n\}$.

Then $x_{N+1} \in \mathcal{D}$ but $x_{N+1} \in \mathbb{R}/[a_k, b_k]$ for all $1 \le k \le n$. $\lightning$.

Since $\{x_{j_k}\}$ is a subsequence of $\{x_j\}$ it converges to x_0 as well (Prop. 2.30).
Since $[a_{k_0}, b_{k_0}]$ is closed, $x_0 \in [a_{k_0}, b_{k_0}]$.
Thus $x_0 \in \mathcal{D}$ as claimed.

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b) Let $\{x_j\}$ be a sequence in $\mathcal{D}$.
By the claim above, there is a subsequence $\{x_{j_k}\}$ in $[a_{k_0}, b_{k_0}]$ for some $1 \le k_0 \le n$.
Since $[a_{k_0}, b_{k_0}]$ is sequentially compact (Theorem 2.36) there is a subsequence of $\{x_{j_k}\}$ converging to an element of $[a_{k_0}, b_{k_0}]$. This element is in $\mathcal{D}$.

c) • Without loss of generality, assume all intervals are disjoint. (i.e. $[a_k, b_k] \cap [a_j, b_j] = \phi$ whenever $k \neq j$.)

Otherwise, there are $n-1$ or fewer closed bounded intervals in $\mathcal{D}$.

• Without loss of generality assume $a_1 < a_2 < \cdots < a_n$. (Otherwise relabel.)

• Take
$$\delta = \min\{a_{k+1} - b_k \mid 1 \leq k \leq n-1\}.$$

Since all intervals are disjoint $\delta > 0$.

Now: for each $1 \leq k \leq n$, consider the function $f_k : [a_k, b_k] \to \mathbb{R}$ with $f_k(x) = f(x)$ for $x \in [a_k, b_k]$.
This f_k is continuous and by Theorem 3.17, f_k is uniformly continuous.
Thus for any $\varepsilon > 0$, $\exists\, \delta_k > 0$ s.t.

$$|f(x) - f(y)| = |f_k(x) - f_k(y)| < \varepsilon$$

whenever $|x - y| < \delta_k$ for $x, y \in [a_k, b_k]$.

Claim: Take $\delta' = \min\{\delta, \delta_1, \ldots, \delta_n\}$.

Then $\delta' > 0$ and

$|f(x) - f(y)| < \varepsilon$ if $|x-y| < \delta'$ $x, y \in \mathcal{D}$.

This is because if $x, y \in \mathcal{D}$ and $|x-y| < \delta' \leq \delta$
This means $x, y \in [a_k, b_k]$ for some $1 \leq k \leq n$.

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d). As before, for each $1 \leq k \leq n$ let
$f_k : [a_k, b_k] \to \mathbb{R}$ be given by

$f_k(x) = f(x)$ for all $x \in [a_k, b_k]$.

By theorem 3.9, the extreme value theorem holds for each f_k, thus there is $x_k^{min}, x_k^{max} \in [a_k, b_k]$ and

$f_k(x_k^{min}) \leq f_k(x) \leq f_k(x_k^{max})$ for all $x \in [a_k, b_k]$

Take $f_- = \min\{ f_k(x_k^{min}) \mid 1 \leq k \leq n \}$
and

$f_+ = \max\{ f_k(x_k^{max}) \mid 1 \leq k \leq n \}$.

Then for any $x \in \mathcal{D}$, $x \in [a_k, b_k]$ for some k and hence

$f_- \leq f_k(x) = f(x) = f_k(x) \leq f_+$