Show all work. You may use any result proven in class, or homework problem, but you must state the results you are using precisely.

(1) a) Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = x^2 + 4 \quad \text{for all } x \in \mathbb{R}.$$ 

Is $f$ uniformly continuous? Justify your answer.

b) Consider the function $g : \mathbb{R} \to \mathbb{R}$ given by

$$g(x) = \frac{1}{x^2 + 4} \quad \text{for all } x \in \mathbb{R}.$$ 

Is $g$ uniformly continuous? Justify your answer.
(2) a) Let $g : \mathbb{R} \to \mathbb{R}$ be differentiable with $g'(x) > 0$ for all $x \in \mathbb{R}$ and moreover, suppose $g(\mathbb{R}) = \mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Define $h : \mathbb{R} \to \mathbb{R}$ by setting

$$h(x) = f(g^{-1}(x)) \quad \text{for all } x \in \mathbb{R}.$$ 

Show that $h$ is differentiable and find $h'(x)$ for all $x \in \mathbb{R}$.

b) Let $x_0 \in \mathbb{R}$ and suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $x_0$. Determine whether the following limit exists:

$$\lim_{x \to x_0} \frac{xf(x_0) - x_0f(x)}{x - x_0}$$

If the limit exists, find its value. If not, explain. In either case, justify your answer.
(3) Consider the function \( f : [0, 1] \rightarrow \mathbb{R} \) defined by

\[
f(x) = \begin{cases} 
  x & \text{if } x \in [0, 1] \text{ is rational.} \\
  0 & \text{if } x \in [0, 1] \text{ is irrational.}
\end{cases}
\]

Prove that \( f \) is not integrable by showing that

\[
\int_0^1 f = 0 \quad \text{and} \quad \overline{\int_0^1} f \geq 1/2.
\]
Let $f : [0, 1] \to \mathbb{R}$ be continuous and monotonically increasing. Let $g : [1, 2] \to \mathbb{R}$ be continuous and monotonically decreasing. Define $h : [0, 2] \to \mathbb{R}$ by setting

$$h(x) = \begin{cases} 
  f(x) & \text{if } x \in [0, 1), \\
  0 & \text{if } x = 1, \\
  g(x) & \text{if } x \in (1, 2].
\end{cases}$$

Prove that $h$ is integrable on $[0, 2]$ by finding an Archimedean sequence for $h$ on $[0, 2]$. Express the integral of $h$ in terms of the integrals of $f$ and $g$. Justify your claims.