1) a) No, \( f(x) = x^2 + 4 \) is not uniformly continuous on \( \mathbb{R} \). Consider two sequences \( u_n \) and \( v_n \) with

\[
\begin{align*}
u_n &= n + \frac{1}{n} \\
\text{and} \quad u_n &= n \quad \text{for all } n \in \mathbb{N}.
\end{align*}
\]

Clearly

\[
\frac{1}{n} \quad \text{and so} \quad \lim_{n \to \infty} u_n - v_n = 0.
\]

Also

\[
f(u_n) = (n + \frac{1}{n})^2 + 4 = n^2 + \frac{1}{n^2} + 6 \quad \text{and} \quad f(v_n) = n^2 + 4.
\]

Thus

\[
f(u_n) - f(v_n) = 2 + \frac{1}{n^2} \quad \text{and so} \quad \lim_{n \to \infty} f(u_n) - f(v_n) = 2 \neq 0.
\]

1) b) Yes \( g(x) = \frac{1}{x^2 + 4} \) is uniformly continuous on \( \mathbb{R} \).

Let \( x, y \in \mathbb{R} \), note that

\[
g(x) - g(y) = \frac{1}{x^2 + 4} - \frac{1}{y^2 + 4} = \frac{(y^2 + 4) - (x^2 + 4)}{(x^2 + 4)(y^2 + 4)} = \frac{y^2 - x^2}{(x^2 + 4)(y^2 + 4)}
\]

Thus

\[
|g(x) - g(y)| = \frac{|x - y| \cdot (|x| + |y|)}{(x^2 + 4)(y^2 + 4)} \leq |x - y| \left( \frac{1}{x^2 + 4} + \frac{1}{y^2 + 4} \right)
\]

Cauchy

\[
\leq |x - y| \left( \frac{1}{2} \frac{(x^2 + 1)}{(x^2 + 1)} + \frac{1}{2} \frac{(y^2 + 1)}{(y^2 + 1)} \right)
\]

Thus for all \( \varepsilon > 0 \) there is \( \delta = \varepsilon > 0 \) s.t.

\[
|g(x) - g(y)| < \varepsilon \quad \text{if} \quad |x - y| < \delta = \varepsilon.
\]
2a) Since \( g'(x) > 0 \) for all \( x \in \mathbb{R} \), we have a result (Corollary 4.21) which shows that \( g \) is strictly increasing. Since \( g \) is strictly increasing with non-zero derivative, we have a result (Theorem 4.11) which shows that \( g^{-1} \) is differentiable. In fact, by Corollary 4.11,

\[
(g^{-1})'(x) = \frac{1}{g'(g^{-1}(x))}.
\]

Since \( f \) and \( g^{-1} \) are both differentiable, the Chain Rule (Theorem 4.14) applies and so \( h = f \circ g^{-1} \) is differentiable. Moreover,

\[
h'(x) = f'(g^{-1}(x))(g^{-1})'(x) = f'(g^{-1}(x)) \cdot \frac{1}{g'(g^{-1}(x))}.
\]

2b) Note that for any \( x \neq x_0 \), the quantity of interest is:

\[
\frac{xf(x_0) - x_0 f(x)}{x - x_0} = \frac{(x-x_0+x_0)f(x_0) - x_0f(x)}{x-x_0}
\]

\[
= f(x_0) + \frac{x_0(f(x_0)-f(x))}{x-x_0}
\]

\[
= f(x_0) - x_0 \frac{(f(x)-f(x_0))}{x-x_0}
\]

Thus the limit exists and

\[
\lim_{x \to x_0} \frac{xf(x_0) - x_0 f(x)}{x - x_0} = f(x_0) - x_0 f'(x_0)
\]

Since \( f \) is differentiable at \( x_0 \), i.e., \( f'(x_0) = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} \).
3) We first prove that \( \underline{\int}_{0}^{1} f = 0 \).

Recall: \( \underline{\int}_{0}^{1} f = \text{sup}_{P} L(f, P) \) ( \( P \) is a partition of \( [0,1] \)).

We prove that \( L(f, P) = 0 \) for all partitions \( P \). The above follows.

Let \( P \) be a partition of \( [0,1] \).

\[ L(f, P) = \sum_{i=1}^{n} m_i (x_i - x_{i-1}) \] where \( m_i = \text{inf}_{x_{i-1}}^{x_i} f(x) \).

Since \( f(x) = 0 \) for all \( x \in [0,1] \), each \( m_i \geq 0 \).

Since irrational numbers are dense, there is \( x^* \in (x_{i-1}, x_i) \cap \mathbb{Q} \).

Thus \( m_i \leq f(x^*) = 0 \) \( \Rightarrow m_i = 0 \) for all \( i \) and hence

\( L(f, P) = 0 \).

We next prove that \( \overline{\int}_{0}^{1} f \geq 1/2 \).

Recall: \( \overline{\int}_{0}^{1} f = \text{inf}_{P} U(f, P) \) ( \( P \) is a partition of \( [0,1] \)).

Let \( g: [0,1] \rightarrow \mathbb{R} \) be the function \( g(x) = x \).

Since \( g \) is increasing, we know \( g \) is integrable on \( [0,1] \).

As proven in class, the regular partitions \( P_n \) are Archimedean for \( g \) and so

\[ \overline{\int}_{0}^{1} g = \underline{\int}_{0}^{1} g = \lim_{n \to \infty} U(g, P_n) = \lim_{n \to \infty} \sum_{i=1}^{n} M_i(g) (x_i - x_{i-1}) \]

\[ = \lim_{n \to \infty} \sum_{i=1}^{n} x_i \frac{1}{n} = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} x_i = \frac{1}{2} \]

\[ U(f, P) = \sum_{i=1}^{n} M_i(f) (x_i - x_{i-1}) = \sum_{i=1}^{n} x_i (x_i - x_{i-1}) = U(g, P) \]

Claim: For all partitions \( P \) of \( [0,1] \),

\[ U(f, P) = U(g, P) \).

In this case, \( \underline{\int}_{0}^{1} f = \underline{\int}_{0}^{1} g = 1/2 \).
there we use that:

\[ M_i(t) = \sup \{ f(x) \mid x \in [x_{i-1}, x_i] \} = x_i \]

\[ f(x) \leq x_i \text{ for all } x \in [x_{i-1}, x_i] \]

Since rational numbers are dense, there is a sequence \( \{x_n\} \subseteq [x_{i-1}, x_i] \) with \( x_n \to x_i \)

Thus \( x_n = f(x_n) \leq M_i(t) \leq x_i \)

4. Since \( f : [0,1] \to \mathbb{R} \) is increasing, the sequence of regular partitions \( \{P_n\} \) is Archimedean for \( t \).

Since \( g : [1,2] \to \mathbb{R} \) is decreasing, the sequence of regular partitions \( \{P_n\} \) is Archimedean for \( f \).

Claim: \( \{P_n\} \) with \( P_n = [0,1] U [1,2] \) is Archimedean for \( h \).

Note: \( U(h, P_n) = U(t, P_{[0,1]}) + U(g, P_{[1,2]}) + \frac{(M_n(h) - M_n(t))}{n} \)

also

\[ L(h, P_n) = L(t, P_{[0,1]}) + L(g, P_{[1,2]}) + \frac{(m_n(h) - m_n(t))}{n} \]

Since \( f, g, \) and \( h \) are bounded,

\[ U(h, P_n) - L(h, P_n) = \left( U(t, P_{[0,1]}) - L(t, P_{[0,1]}) \right) + \left( U(g, P_{[1,2]}) - L(g, P_{[1,2]}) \right) + \frac{(M_n(h) - M_n(t))}{n} \]

\[ + \frac{(m_n(h) - m_n(t))}{n} \]

\[ \Rightarrow \lim_{n \to \infty} (U(h, P_n) - L(h, P_n)) = 0 \quad \text{since} \quad |R_n| \leq \frac{8M}{n} \]
\[ \sum_{n=0}^{2} \lim_{n \to \infty} U(h/n) = \lim_{n \to \infty} \left( U(f, n \frac{3}{2}) + U(g, n \frac{1}{2}) + \frac{M_{n}(h) - M_{n}(f)}{n} + \frac{M_{n}(h) - M_{n}(g)}{n} \right) \\
= \sum_{n=0}^{2} f + \sum_{n=0}^{2} g + 0. \]