Exam 2: MakeUp Key

1. Let \( f : (a, b) \to \mathbb{R} \) be differentiable and \( f' : (a, b) \to \mathbb{R} \) be continuous. Let \( c, d \in (a, b) \). Show that for each \( \varepsilon > 0 \), there is \( \delta > 0 \) such that

\[
|f(x) - f(y) - f'(y)(x-y)| < \varepsilon
\]

whenever \( x, y \in [c, d] \) satisfy

\[
|x-y| < \delta.
\]

Proof:
Since \( f : (a, b) \to \mathbb{R} \) is differentiable, \( f' : (a, b) \to \mathbb{R} \) is also continuous. In this case, for each \( c, d \in (a, b) \), \( f : [c, d] \to \mathbb{R} \) is continuous and also uniformly continuous. Thus for every \( \varepsilon' > 0 \), there is \( \delta > 0 \) for which

\[
|f(x) - f(y)| < \varepsilon'/2
\]

whenever \( x, y \in [c, d] \) satisfy

\[
|x-y| < \delta_1.
\]
Since \( f'(a,b) \to \mathbb{R} \) is continuous,
\( f': \mathbb{R}^2 \to \mathbb{R} \) is continuous as well.
Moreover, \( f': \mathbb{R}^2 \to \mathbb{R} \) is bounded.
Thus there is \( M > 0 \) for which
\[
|f'(x)| \leq M \quad \text{for all } x \in \mathbb{R}^2.
\]

Now let \( \varepsilon > 0 \). Take \( \delta_2 > 0 \) such that \( \delta_2 = \frac{\varepsilon}{2M} \)
with \( \delta = \min(\delta_1, \delta_2) > 0 \) and \( x, y \in \mathbb{R}^2 \)
\[
|x - y| < \delta \quad \text{means that}
\]
\[
|f(x) - f(y) - f'(y)(x - y)|
\]
\[
\leq |f(x) - f(y)| + |f'(y)| |x - y|
\]
\[
\leq \frac{\varepsilon}{2} + M \cdot \delta
\]
\[
\leq \frac{\varepsilon}{2} + M \cdot \delta_2
\]
\[
\leq \varepsilon
\]
a) Let \( f : \mathbb{R} \to \mathbb{R} \) be differentiable. Determine whether or not the limit
\[
\lim_{x \to 0} \frac{f(x^2) - f(0)}{x}
\]
deexists.

b) Let \( g : \mathbb{R} \to \mathbb{R} \) be given by
\[
g(x) = \begin{cases} 
  x - x^2 & \text{if } x \in \mathbb{R} \\
  x + x^2 & \text{if } x \in (\mathbb{R} \setminus \mathbb{Q})
\end{cases}
\]
Find \( g'(0) \). Discuss behavior of \( g \) in a neighborhood of 0.

2a) Proof:
Set \( f : \mathbb{R} \to \mathbb{R} \) to be
\[
g(x) = x^2
\]
we know that \( g \) is differentiable and
\[
g'(x) = 2x
\]
In this case, \( h : \mathbb{R} \to \mathbb{R} \) defined by setting
\[
h(x) = (f \circ g)(x)
\]
is also differentiable by the chain rule.
In this case,

\[ h'(a) = \lim_{x \to a} \frac{h(x) - h(a)}{x - a} \exists \]

\[ = \lim_{x \to 0} \frac{(f \circ g)(x) - (f \circ g)(0)}{x} \]

\[ = \lim_{x \to 0} \frac{f(g(x)) - f(0)}{x} \]

By the chain rule, we also know that

\[ h'(x) = (f \circ g)'(x) = f'(g(x)) \cdot g'(x) \]

\[ = 2x \cdot f'(g(x)^2) \]

\[ \Rightarrow h'(10) = 2 \cdot 10 \cdot f'(10^2) = 0. \]

b)

\[ g'(a) = \lim_{x \to a} \frac{g(x) - g(a)}{x - a} \]

Since \( a \in \mathbb{R}^2 \), \( g(a) = 0 - 0^2 = 0 \).

\[ g(x) = \left\{ \begin{array}{ll}
\frac{x - x^2}{x} & \text{if } x \in \mathbb{Q} \\
\frac{x + x^2}{x} & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}
\end{array} \right. \]

\[ = \left\{ \begin{array}{ll}
1 - x & \text{if } x \in \mathbb{Q} \cup \{-\infty, 0, \infty\} \\
1 + x & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}
\end{array} \right. \]
In this case,

\[ \lim_{x \to 0} \frac{g(x)}{x} = 1 \quad \text{is} \quad g'(0) = 1 > 0. \]

From calculus, one might expect that there is a neighborhood at 0 on which \( f \) is increasing, but this is not the case.

**Note that:**

If \( x \in \mathbb{R} \setminus \{ 0 \} \) and \( y \in \mathbb{R} \), then

\[ g(y) - g(x) = (y - y^2) - (x + x^2) \]

\[ = (y - x) - (x^2 + y^2) \]

Let \( n \in \mathbb{N} \). Take \( y_n = \frac{1}{2n} \). Take \( x_n \in \left( \frac{1}{2n}, \frac{y_n}{2} \right) \) and

\[ x_n < y_n. \]

Then

\[ g(y_n) - g(x_n) = (y_n - x_n) - (x_n^2 + y_n^2) \]

\[ \leq \left( \frac{1}{2n^2} - \frac{1}{2n} \right) \]

\[ \leq \frac{1}{2n} - \left( x_n^2 + \frac{1}{2an} \right) \]

\[ = -x_n^2 < 0 \]

Is decreasing along this sequence...
3a) Let $\mathcal{R}_n$ be a sequence of partitions of $[a,b]$. We say that $\mathcal{R}_n$ is a sequence of refinements of $\mathcal{R}_m$ if: For each $n \geq m$, $\mathcal{R}_n$ is a refinement of $\mathcal{R}_m$. Let $f: [a,b] \to \mathbb{R}$ be integrable. Let $\mathcal{R}_n$ be an Archimedean sequence for $f$ on $[a,b]$. Prove that every sequence of refinements $\mathcal{R}_n$ of $\mathcal{R}_m$ is also Archimedean for $f$ on $[a,b]$.

b) Consider $f: [2,4] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 
3x - 1 & \text{if } 2 \leq x \leq 3, \\
4 & \text{if } 3 < x \leq 4.
\end{cases}$$

Show that $f$ is integrable and find the value of $\int_{2}^{4} f(x) \, dx$.

3b) Proof. Since $\mathcal{R}_n$ is Archimedean for $f$ on $[a,b]$, we know that

$$\lim_{n \to \infty} (U(f, \mathcal{R}_n) - L(f, \mathcal{R}_n)) = 0.$$ 

If $\mathcal{R}_n$ is a sequence of refinements of $\mathcal{R}_m$, then...
For each $n \in \mathbb{N}$,

$$U(f, p_n) \leq L(f, \bar{p}_n) \quad \text{and} \quad U(f, \bar{p}_n) \leq L(f, p_n)$$

by the Riemann lemma.

In this case, for each $n \in \mathbb{N}$,

$$0 \leq U(f, \bar{p}_n) - L(f, p_n) \leq U(f, \bar{p}_n) - L(f, p_n)$$

Thus, by the Riemann lemma,

$$\lim_{n \to \infty} (U(f, \bar{p}_n) - L(f, p_n)) = 0$$

using (**) This proves that $\int_{a \, b}$ is Archimedean for $f$ on $[a \, b]$.

b) For each $n \in \mathbb{N}$, let

- $p_n$ be a regular partition of $[2, 3]$ and
- $q_n$ be a regular partition of $[3, 4]$.

That

$$p_n = \frac{1}{n} \times \frac{3}{n} \quad \text{with} \quad y_j = 2 + \frac{j}{n} \quad \text{for} \quad 0 \leq j \leq n$$

and

$$q_n = \frac{1}{n} \times \frac{1}{n} \quad \text{with} \quad y_j = 3 + \frac{j}{n} \quad \text{for} \quad 0 \leq j \leq n$$

Take $\bar{p}_n = p_n \cup q_n$ which is a partition of $[2, 4]$. 
Note that
\[ U(t, \mathbf{p}_n) = U(t, \mathbf{p}_0) + U(t, \mathbf{q}_n) \]
and
\[ U(t, \mathbf{p}_n) = U(t, \mathbf{p}_0) + U(t, \mathbf{q}_n) \]
We show the 1st, the 2nd is similar.

Let \( \mathbf{p}_n = \sum_{j=0}^{n} \mathbf{z}_j \) where \( \mathbf{z}_j = \sum_{y_j \in \mathcal{J}} \mathbf{y}_j \)
with \( \mathcal{J} = \{ x_1^2 \} \)

Then
\[ U(t, \mathbf{p}_n) = \sum_{j=0}^{n} M_j(t) (\mathbf{x}_j - \mathbf{x}_{j-1}) \]
\[ = \sum_{j=1}^{n} M_j(t) (\mathbf{x}_j - \mathbf{x}_{j-1}) \]
\[ + \sum_{j=0}^{n} M_j(t) (\mathbf{y}_j - \mathbf{y}_{j-1}) \]
\[ = U(t, \mathbf{p}_0) + U(t, \mathbf{q}_n) \]
The rest follows similarly.

Then
\[ 0 \leq U(t, \mathbf{p}_n) - U(t, \mathbf{p}_0) \]
\[ = \left( U(t, \mathbf{p}_n) - U(t, \mathbf{p}_0) \right) + \left( U(t, \mathbf{q}_n) - U(t, \mathbf{q}_0) \right) \]
\[ u(t, q_0) - u(t, p_n) = \sum_{i=1}^{n} (M_i(t) - m_j(t)) (x_i - x_{i-1}) = \frac{1}{n} \sum_{i=1}^{n} (x_i - x_{i-1}) \cdot 1 = \frac{1}{n} \cdot \sum_{i=1}^{n} 1 = \frac{1}{n} \cdot \sum_{i=1}^{n} 1 = \frac{1}{n} \]

and
\[ u(t, q_n) - u(t, q_n) = \sum_{i=1}^{n} (M_i(t) - m_j(t)) (y_i - y_{i-1}) = (M_1(t) - m_1(t)) \cdot \frac{1}{n} + \sum_{i=2}^{n} \frac{1}{n} \]
\[ = (4 - 2) \cdot \frac{1}{n} = \frac{2}{n}. \]

Thus
\[ \lim_{n \to \infty} (u(t, q_0) - u(t, q_n)) = \lim_{n \to \infty} (u(t, p_n) - u(t, q_n)) \]
\[ + \lim_{n \to \infty} (u(t, q_n) - u(t, q_n)) \]
\[ = \lim_{n \to \infty} \frac{1}{n} + \lim_{n \to \infty} \frac{1}{n} \cdot 0 = 0 + 0 = 0. \]

Thus \( g \) is Archimedean for \( f \) on \( t \geq 1 \).

Moreover
\[ \sum_{j=1}^{n} f = \lim_{n \to \infty} u(t, p_n) = \lim_{n \to \infty} (u(t, p_n) + u(t, q_n)) \]
\[ = \lim_{n \to \infty} \left( \sum_{j=1}^{n} (x_j - x_{j-1}) + \sum_{j=1}^{n} y_j \right) = \frac{1}{n} \left( \sum_{j=1}^{n} (x_j - x_{j-1}) + 4 \sum_{j=1}^{n} y_j \right) = 5 \frac{1}{n} \]
\[ \text{use that} \]
\[ \sum_{j=1}^{n} y_j = n \frac{1}{2} \left( \sum_{j=1}^{n} (x_j - x_{j-1}) \right) + 4 \sum_{j=1}^{n} y_j = 5 \frac{1}{2} \]
4) Let \( f: \mathbb{R} \to \mathbb{R} \) be continuous.
Let \( a, b: \mathbb{R} \to \mathbb{R} \) be differentiable.
Define \( F: \mathbb{R} \to \mathbb{R} \) by setting:
\[
F(x) = \int_{a(x)}^{b(x)} f(t) \, dt.
\]

Show that \( F \) is differentiable and evaluate \( F'(x) \).

**Proof:** Since \( f: \mathbb{R} \to \mathbb{R} \) is continuous,
for each define \( G: \mathbb{R} \to \mathbb{R} \) by setting
\[
G(x) = \int_{c}^{x} f(t) \, dt.
\]

This is well defined since \( f: [a, x] \to \mathbb{R} \) is continuous for each \( x \geq 0 \) and \( f: [x, b] \to \mathbb{R} \) is continuous for each \( x \leq 0 \).

Now that,
\[
F(x) = \int_{a(x)}^{b(x)} f(t) \, dt = \int_{b(x)}^{c} f(t) \, dt + \int_{c}^{a(x)} f(t) \, dt
\]
\[
= G(b(x)) - G(a(x))
\]
By the Fundamental Theorem of calculus and the chain rule, both $g \circ a$ and $g \circ b$ are differentiable.

Moreover,

$$F'(x) = (g \circ b)'(x) - (g \circ a)'(x)$$

$$= G'(b(x)) \cdot b'(x) - G'(a(x)) \cdot a'(x)$$

$$= f(b(x)) \cdot b'(x) - f(a(x)) \cdot a'(x).$$