MATH 464:
TEST 1

SPRING 2016

Name	Key
I.D. Number	

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total	80	

Do all of the following problems. An answer alone will receive no credit. Justify all your claims.

(1) Flip a fair coin 4 times.

a) Write out the sample space.

$$\Omega = \left\{ \begin{array}{llll} HHHH & HHHT & HTTH & THTT \\ THHH & TTHH & HTHT & TTHT \\ HTHH & THTH & HHTT & TTTH \\ HHTH & THHT & HTTT & TTTT \end{array} \right\}$$

b) What is the likelihood that you get at most two tails?

$A = \{ \underline{HHHH}, \underline{THHH}, \underline{HTHH}, \underline{HHTH}, \underline{HHHT}, \underline{TTHH}, THTH, \underline{THHT}, HTTH, HTHT, \underline{HHTT} \}$

$$P(A) = \frac{11}{16}$$

c) What is the likelihood that you get at least two consecutive heads given that you get at least two heads?

B = at least two heads – same event as above

A = at least two consecutive heads – underlined above

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{\frac{8}{16}}{\frac{11}{16}} = \frac{8}{11}$$

(2) You have three alarm clocks that will ring on any given morning with probabilities 0.6, 0.75, and 0.85, respectively. For an important exam, you set all three alarm clocks.

a) What is the probability that you will be awakened by at least one of your alarm clocks?

Let A be the event that the 1st clock rings.
Let B be the event that the 2nd clock rings
Let C be the event that the 3rd clock rings

We want $D = A \cup B \cup C$

$$P(D) = 1 - P(D^c)$$
$$= 1 - P(A^c \cap B^c \cap C^c)$$
independent
$$= 1 - P(A^c) \cdot P(B^c) \cdot P(C^c)$$
$$= 1 - (1-.6)(1-.75)(1-.8$$
$$= 1 - (.4)(.25)(.15)$$
$$= 1 - 0.015 = .985$$

$\boxed{98.5\%}$

b) What is the probability that exactly two alarm clocks will ring?

$$E = (A \cap B \cap C^c) \cup (A \cap B^c \cap C) \cup (A^c \cap B \cap C)$$ ← disjoint

$$P(E) = P(A \cap B \cap C^c) + P(A \cap B^c \cap C) + P(A^c \cap B \cap C)$$

← independent

$$= P(A) \cdot P(B) \cdot P(C^c) + P(A) \cdot P(B^c) \cdot P(C) + P(A^c) \cdot P(B) \cdot P(C)$$

$$= (0.6)(0.75)(1-0.85) + (0.6)(1-0.75)(0.85) + (1-.6)(0.75)(0.8$$

$$= 0.0675 + 0.1275 + 0.255$$

$$= 0.45$$

45%

(3) Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $A, B \in \mathcal{F}$. Prove that the event that *exactly* one of A and B occur has probability:

$$P(A) + P(B) - 2P(A \cap B).$$

Let C be the event that exactly one of A and B occur.

$C = (A \cap B^c) \cup (A^c \cap B)$ ← a disjoint union

$P(C) = P(A \cap B^c) + P(A^c \cap B)$

$= P(A \setminus B) + P(B \setminus A)$ ← def. of $A \setminus B$ and $B \setminus A$

$= (P(A) - P(A \cap B)) + (P(B) - P(B \cap A))$

↑ properties of probability measures

$= P(A) + P(B) - 2P(A \cap B).$

(4) You have 10 dimes and 10 pennies. You put these coins in two boxes: the first box contains 6 dimes and 2 pennies and the second box contained 4 dimes and 8 pennies. You fairly choose a box at random and then fairly pick a coin out of that box at random.

a) What is the probability that the coin you selected is a dime?

Let A be the event that you selected a dime.
Let B_1 be the event that you chose the 1st box.
Let B_2 be the event that you chose the 2nd box.
Clearly $\Omega = B_1 \cup B_2$ is a partition. Thus

$$P(A) = P(A|B_1)\cdot P(B_1) + P(A|B_2)\cdot P(B_2) = 6/8 \cdot \tfrac{1}{2} + 4/12 \cdot \tfrac{1}{2} = \frac{13}{24} \cong 54.167\%$$

b) The next day you forgot what box you chose, but you remember that the selected coin was a dime. What is the probability that you took the dime from the first box?

$$P(B_1|A) = \frac{P(A|B_1)\cdot P(B_1)}{P(A)} = \frac{\frac{6}{8}\cdot\frac{1}{2}}{\frac{13}{24}} = \frac{9}{13}$$

$$\cong 69.23\%$$

(5) Roll a fair 4-sided die twice. Let X be the discrete random variable that takes the maximum of the two rolls. Let $Y = X^2 - 4X + 3$. Find f_X, f_Y, $E(X)$, and $E(Y)$.

1st roll

X	1	2	3	4
1	1	2	3	4
2	2	2	3	4
3	3	3	3	4
4	4	4	4	4

2nd roll

x	1	2	3	4
$f_X(x)$	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{5}{16}$	$\frac{7}{16}$

$$E(X) = 1 \cdot f_X(1) + 2 \cdot f_X(2) + 3 \cdot f_X(3) + 4 \cdot f_X(4)$$
$$= \frac{1}{16} + \frac{6}{16} + \frac{15}{16} + \frac{28}{16}$$
$$= \frac{50}{16} = 3.125$$

$$Y = X^2 - 4X + 3$$
$$= (X-3)(X-1)$$

$X=1 \Rightarrow Y=0$

$X=2 \Rightarrow Y=(-1)(1) = -1$

$X=3 \Rightarrow Y=0$

$X=4 \Rightarrow Y=(1)(3)=3$

y	-1	0	3
$f_Y(y)$	$\frac{3}{16}$	$\frac{6}{16}$	$\frac{7}{16}$

$$E(Y) = -1 \cdot f_Y(y) + 0 \cdot f_Y(0) + 3 \cdot f_Y(3)$$
$$= -\frac{3}{16} + \frac{21}{16}$$
$$= \frac{18}{16} = 1.125$$

(6) Let X be a discrete random variable. Suppose we know that

$\mathbf{x}$	-1	0	1	2	3
$\mathbf{f_X(x)}$	0.15	0.1	0.25	?	0.2

a) Note that only one probability is missing. Assuming that X has no other values in its range, find the missing probability.

$P(X = 2) =$ $.3$

$1 - 0.15 - 0.1 - 0.25 - 0.2 = .3$

b) Find

$P(0 < X < 3) =$ $P(X=1) + P(X=2)$

$= 0.25 + 0.3 = .55$

c) Find

$P(0 < X < 1) =$ $P(\phi) = 0$

d) Find

$P(X \leq 0 \,|\, |X| = 1) =$ $\dfrac{P(X=-1)}{P(X=-1)+P(X=1)} = \dfrac{0.15}{0.15+0.25}$

$= \dfrac{0.15}{0.4}$

$= .375$

e) Find

$E(X^2) =$

$E(X^2) = (-1)^2 \cdot 0.15 + 0^2 \cdot .1 + 1^2 \cdot 0.25 + 2^2 \cdot .3 + 3^2 \cdot .2$

$= 0.15 + 0.25 + 1.2 + 1.8$

$= 3.4$

(7) a) Let X be a Poisson random variable with parameter $\lambda > 0$. Find

$$E(e^X).$$

For full credit you must simplify the resulting sum.

$$E(e^X) = \sum_x e^x P(X=x) = \sum_{k=0}^{\infty} e^k \cdot P(X=k)$$

$$= \sum_{k=0}^{\infty} e^k \cdot e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \cdot \sum_{k=0}^{\infty} \frac{(e\lambda)^k}{k!} = e^{-\lambda} \cdot e^{e\lambda}$$

$$= e^{\lambda(e-1)}$$

b) Let X be a geometric random variable with parameter $p > 0$. Consider $Y = \cos(\pi X)$. Find the probability mass function associated to Y. In fact, write f_Y as a table and for full credit, simplify all sums.

Since X is geometric, it has range $\{1, 2, 3, \ldots\}$

y	-1	1
$f_Y(y)$	$\frac{1}{2-p}$	$\frac{1-p}{2-p}$

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$$Y = \begin{cases} 1 & k \geq 1 \text{ even} \\ -1 & k \geq 1 \text{ odd} \end{cases}$$

$$P(Y=1) = \sum_{n\geq 1} P(X=2n) = \sum_{n \geq 1} p(1-p)^{2n-1}$$

$$= p(1-p) \sum_{n \geq 1} (1-p)^{2(n-1)}$$

$$= p(1-p) \cdot \frac{1}{1-(1-p)^2}$$

$$= \frac{p(1-p)}{1-1+2p-p^2} = \frac{1-p}{2-p}$$

$$P(Y=-1) = \sum_{n \geq 0} P(X=2n+1)$$

$$= \sum_{n \geq 0} p(1-p)^{2n}$$

$$= p \cdot \frac{1}{1-(1-p)^2} = \frac{1}{2-p}$$

(8) Let X be a discrete random variable. Suppose X has mean -2 and variance 3. Let $Y = X^2 + 5$ and suppose Y has variance 1.

a) What is the mean of Y?

$$3 = Var(X) = E((X-\mu_X)^2) = E(X^2) - \mu_X^2$$
$$= E(X^2) - (-2)^2$$

So

$$\mu_Y = E(Y) = E(X^2) + 5 = (3+4) + 5 = 12$$

b) What is the 4th moment of X?

$$1 = Var(Y) = E((Y-\mu_Y)^2) = E((X^2-7)^2)$$
$$= E(X^4) - 14\,E(X^2) + 49$$
$$= E(X^4) - 14(3+4) + 49$$

$$\Rightarrow \quad E(X^4) = 1 + 98 - 49$$
$$= 50$$