MATH 464:
TEST 2

SPRING 2016

Name	Key
I.D. Number	

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
Total	70	

Do all of the following problems. An answer alone will receive no credit. Justify all your claims.

(1) A box has 6 red balls, 3 green balls, and 4 white balls - all of equal size and weight. You randomly select a ball from the box and then flip a coin (one with probability p for heads):
i) until you get heads, if the ball you selected was red,
ii) until you get tails, if the ball you selected was green, and
iii) three times if the ball was white.
Let X be the discrete random variable which counts the total number of flips in each outcome. Find $E(X)$.

Let B_1 be the event that you select a red ball
Let B_2 " " " a green ball.
Let B_3 " " " a white ball.

Since these events form a partition,

$$E(X) = E(X|B_1)\cdot P(B_1) + E(X|B_2)\cdot P(B_2) + E(X|B_3)\cdot P(B_3)$$

$$= \frac{1}{p}\cdot\frac{6}{13} + \frac{1}{1-p}\cdot\frac{3}{13} + 3\cdot\frac{4}{13}$$

$$= \frac{6+9p-12p^2}{13p(1-p)}$$

(2) You are dealt 5 cards from a standard deck. You keep careful track of the order of the cards you are dealt.

a) What is the probability that the cards you are dealt alternate in color?

Since the cards are ordered: $|\Omega| = P_{52,5} = \dfrac{(52)!}{(52-5)!}$

$= 311,875,200$

There are only 2 colors: R = Red and B = Black

When they alternate:

RBRBR ~ $26 \cdot 26 \cdot 25 \cdot 25 \cdot 24$

or

BRBRB ~ $26 \cdot 26 \cdot 25 \cdot 25 \cdot 24$

$$\text{Prob} = \frac{2 \cdot 26 \cdot 26 \cdot 25 \cdot 25 \cdot 24}{P_{52,5}} \cong 6.5\%$$

b) What is the probability you miss a flush by exactly one card?

To miss a flush: Pick a suit - there are 4 options.

Label these cards S.

To miss by 1, there must be a card not in this suit. Label it by NS.

Since everything is ordered, there are 5 options.

$\boxed{S}\,\boxed{S}\,\boxed{S}\,\boxed{S}\,\boxed{NS}$ ~ $13 \cdot 12 \cdot 11 \cdot 10 \cdot 39$

$\boxed{S}\,\boxed{S}\,\boxed{S}\,\boxed{NS}\,\boxed{S}$ "

$\boxed{S}\,\boxed{S}\,\boxed{NS}\,\boxed{S}\,\boxed{S}$ "

$\boxed{S}\,\boxed{NS}\,\boxed{S}\,\boxed{S}\,\boxed{S}$ "

$\boxed{NS}\,\boxed{S}\,\boxed{S}\,\boxed{S}\,\boxed{S}$ "

$$\text{Prob} = \frac{4 \cdot 5 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 39}{P_{52,5}} \cong 4.3\%$$

(3) I have 24 brownies and 3 friends.

a) How many ways are there for my friends and I to share these brownies with no constraints?

This is the same as r identical objects in n urns.

Here $r = 24$ brownies and $n = 4$ 3 friends and I

$$\# \text{ of ways} = \frac{(r+n-1)!}{r!(n-1)!} = \frac{27!}{24!\,3!} = \frac{27 \cdot 26 \cdot 25}{3 \cdot 2} = 2925$$

b) How many ways are there for us to share the brownies if I insist that my best friend (one of the three) and I each get at least three (with no other constraints)?

First, we fill the constraint.

Give 3 brownies to me and my friend.

In this case, there are only 18 brownies left.

$$\# \text{ of ways} = \frac{(18+4-1)!}{18!(4-1)!} = \frac{21 \cdot 20 \cdot 19}{3 \cdot 2} = 1330$$

(4) Roll a fair 6-sided die. If the value on the die is even, flip a coin (one with probability p for heads) once. If the value on the die is odd, flip a coin (one with probability p for heads) twice. Let X count the number of heads. Let Y be the value on the die.

a) Write a table describing $f_{X,Y}$, f_X, f_Y.

$X \backslash Y$	1	2	3	4	5	6
0	$\frac{1}{4}\cdot\frac{1}{6}$	$\frac{1}{2}\cdot\frac{1}{6}$	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{1}{24}$	$\frac{1}{12}$
1	$\frac{1}{2}\cdot\frac{1}{6}$	$\frac{1}{2}\cdot\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$
2	$\frac{1}{4}\cdot\frac{1}{6}$	0	$\frac{1}{24}$	0	$\frac{1}{24}$	0

$f_{X,Y}$

X	0	1	2
f_X	$\frac{9}{24}$	$\frac{12}{24}$	$\frac{3}{24}$
	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{1}{8}$

Y	1	2	3	4	5	6
f_Y	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

even $\{H, T\}$

odd $\{H,H \;\; H,T \;\; T,H \;\; T,T\}$

b) Based on the results above, determine whether X and Y are independent. Explain.

No, not independent!

$$f_{X,Y}(2,2) = 0 \quad \text{but} \quad f_X(2)\cdot f_Y(2) = \frac{1}{8}\cdot\frac{1}{6} = \frac{1}{48}$$

c) Find $E(X)$.

$$E(X) = 0\cdot f_X(0) + 1\cdot f_X(1) + 2\cdot f_X(2)$$
$$= 0 + 1\cdot\frac{1}{2} + 2\cdot\frac{1}{8}$$
$$= 3/4$$

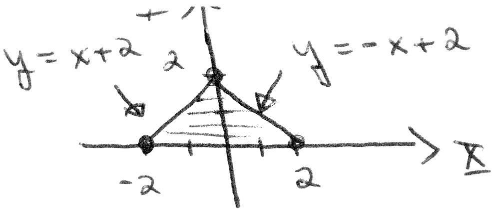
(5) Let X be a Poisson random variable with parameter $\lambda > 0$. Let Y be a geometric random variable with parameter $p > 0$. If X and Y are independent,

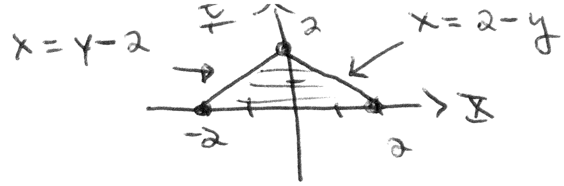
a) Find an expression for
$$P(X+Y=3)$$
Write out each term in your answer.

$$P(X+Y=3) = \sum_{\substack{(x,y):\\ x+y=3}} f_{X,Y}(x,y) = f_{X,Y}(0,3) + f_{X,Y}(1,2) + f_{X,Y}(2,1)$$
$$= e^{-\lambda} p(1-p)^2 + e^{-\lambda} \cdot \lambda p(1-p) + e^{-\lambda}\frac{\lambda^2}{2!} p$$
$$= e^{-\lambda} p\left((1-p)^2 + \lambda(1-p) + \frac{\lambda^2}{2}\right)$$

b) For any integer $n \geq 1$, find an expression for
$$P(X+Y \leq n)$$
You may leave your answer as a sum.

$$P(X+Y \leq n) = \sum_{\substack{(x,y):\\ x+y \leq n}} f_{X,Y}(x,y)$$
$$= \sum_{x=0}^{n-1} \sum_{y=1}^{n-x} f_X(x) \cdot f_Y(y)$$
$$= \sum_{x=0}^{n-1} \sum_{y=1}^{n-x} \left(e^{-\lambda} \frac{\lambda^x}{x!}\right)\left(p(1-p)^{y-1}\right)$$



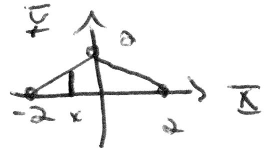


(6) Let X and Y be jointly continuous random variables that are uniformly distributed on the triangle with endpoints $(-2,0)$, $(0,2)$, and $(2,0)$.

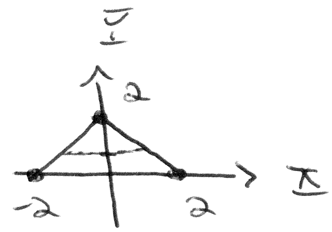
a) Find the marginal distributions f_X and f_Y.

Since the area of the triangle is 4

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{4} & (x,y) \text{ in triangle} \\ 0 & \text{otherwise} \end{cases}$$



$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \begin{cases} \frac{1}{4}(x+2) & -2 \le x \le 0 \\ \frac{1}{4}(-x+2) & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$



$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \begin{cases} \frac{1}{4}[(2-y)-(y-2)] = 1 - \frac{1}{2}y & 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

b) Based on your work in part a), determine whether X and Y are independent. Explain.

No, X and Y are not independent.

$f_X(x) \cdot f_Y(y)$ is not constant!

c) Consider the random variable $Z = X + Y$. Find the mean and variance of Z.

$$\boxed{u = x+y}$$

$$\mu = E(Z) = \int_0^2 \int_{y-2}^{2-y} (x+y) f_{X,Y}(x,y)\,dx\,dy = \frac{1}{4}\int_0^2 \int_{2y-2}^{2} u\,du\,dy$$

$$= \frac{1}{8}\int_0^2 2^2 - 2^2(y-1)^2\,dy \qquad \boxed{w = y-1}$$

$$= 1 - \frac{1}{2}\int_{-1}^{1} w^2\,dw = \frac{2}{3}$$

$$\mathrm{Var}(Z) = E(Z^2) - E(Z)^2 = 4/3 - \frac{4}{9} = \frac{2}{3}$$

$$\boxed{u = x+y}$$

$$E(Z^2) = \int_0^2 \int_{y-2}^{2-y} (x+y)^2 f_{X,Y}(x,y)\,dx\,dy = \frac{1}{4}\int_0^2 \int_{2y-2}^{2} u^2\,du\,dy$$

$$= \frac{1}{12}\int_0^2 2^3 - 2^3(y-1)^3\,dy \qquad \boxed{w = y-1}$$

$$= \frac{8}{12}\cdot 2 - \frac{8}{12}\cancel{\int_{-1}^{1} w^3\,dw}^{\,0} = \frac{4}{3}$$

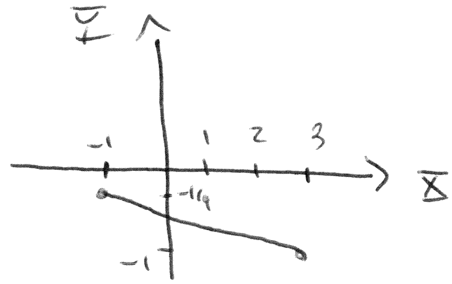
(7) Let X be a continuous random variable with uniform distribution on $[-1, 3]$. Consider the new random variable

$$Y = \frac{1}{2X-7}.$$

a) Graph Y on the range of X.

$$f_X(x) = \begin{cases} \frac{1}{4} & -1 \le x \le 3 \\ 0 & \text{otherwise.} \end{cases}$$

$$F_X(x) = \begin{cases} 0 & x \le -1 \\ \frac{1}{4}(x+1) & -1 \le x \le 3 \\ 1 & x \ge 3 \end{cases}$$



b) Find the cdf, $F_Y(y)$, of Y.

$$F_Y(y) = \begin{cases} 0 & y \le -1 \\ \frac{-1-y}{8y} & -1 \le y \le -1/9 \\ 1 & y \ge -1/9 \end{cases}$$

$$\begin{aligned} F_Y(y) &= P(Y \le y) \\ &= P\left(\frac{1}{2X-7} \le y\right) \\ &= P(1 \ge y(2X-7)) \\ &= P(1+7y \ge 2yX) \\ &= P\left(X \ge \frac{1+7y}{2y}\right) \\ &= 1 - F_X\left(\frac{1+7y}{2y}\right) \\ &= 1 - \frac{1}{4}\left(\frac{1+7y}{2y} + 1\right) \\ &= \frac{8y}{8y} - \frac{1+7y+2y}{8y} \\ &= \frac{-1-y}{8y} \end{aligned}$$

c) Find the pdf, $f_Y(y)$, of Y.

$$f_Y(y) = \begin{cases} 0 & y \le -1 \\ \frac{1}{8y^2} & -1 \le y \le -\frac{1}{9} \\ 0 & y \ge -1/9 \end{cases}$$

$$\frac{d}{dy}\left(\frac{-1-y}{8y}\right) = \frac{8y(-1)+(1+y)8}{64y^2} = \frac{1}{8y^2}$$

d) Is Y uniformly distributed? If so, on what interval?

No!