MATH 464:
TEST 2
MAKE UP

SPRING 2016

Name	Key
I.D. Number	

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
Total	70	

Rules to the Make-Up:

Here are the rules to the make-up. You have two choices. Either you turn in the make-up, or you do not. If you turn in the make-up, it will be graded, and you will receive a new grade for Exam 2. That new grade will be the average of your grade on Exam 2 and the Make-Up to Exam 2. If you do not do the Make-Up, your grade on Exam 2 remains the same. The Make-Up is due on **Tuesday, April 26, 2016**.

Do all of the following problems. An answer alone will receive no credit. Justify all your claims.

(1) Roll a fair, 4-sided die N times where N is a Poisson random variable with parameter $\lambda > 0$. Let X be the number of 3's rolled in this experiment. Find $E(X)$.

For any $n \geq 0$, let

$B_n = \{\omega \in \Omega : N(\omega) = n\}$ - the event that you roll n times.

Then

$$E(X) = \sum_{n=0}^{\infty} E(X \mid B_n) \cdot P(B_n)$$

Given that you rolled $n \geq 0$ times, X is a binomial random variable with parameters n and $p = \frac{1}{4}$.

Thus $E(X \mid B_n) = np = \frac{n}{4}$

Thus

$$E(X) = \sum_{n=0}^{\infty} E(X \mid B_n) \cdot P(B_n)$$

$$= \sum_{n=0}^{\infty} \frac{n}{4} \cdot e^{-\lambda} \cdot \frac{\lambda^n}{n!}$$

$$= \frac{\lambda e^{-\lambda}}{4} \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!} = \frac{\lambda}{4}$$

(2) You are dealt 5 cards from a standard deck. You keep careful track of the order of the cards you are dealt. For both questions below, give an exact answer and a decimal approximation accurate to the nearest one-hundredth.

a) What is the probability that you get one ace and the rest are face cards? (Face cards are the jack, the queen, and the king.)

$|\mathcal{U}| = P_{52,5} = 311{,}875{,}200$

There are 4 aces and

$3 \cdot 4 = 12$ face cards.

$$\begin{array}{ccccc} \boxed{A} & \boxed{F} & \boxed{F} & \boxed{F} & \boxed{F} \\ \boxed{F} & \boxed{A} & \boxed{F} & \boxed{F} & \boxed{F} \\ \boxed{F} & \boxed{F} & \boxed{A} & \boxed{F} & \boxed{F} \\ \boxed{F} & \boxed{F} & \boxed{F} & \boxed{A} & \boxed{F} \\ \boxed{F} & \boxed{F} & \boxed{F} & \boxed{F} & \boxed{A} \end{array}$$

$$\text{Prob} = \frac{4 \cdot 5 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{P_{52,5}} \cong 0.076\%$$

b) What is the probability that you have a pair of tens and a pair of threes (and no better)?

To get 2 pair of 10's and 3's:

$$\left.\begin{array}{ccccc} \boxed{10} & \boxed{10} & \boxed{3} & \boxed{3} & \boxed{X} \\ \boxed{10} & \boxed{3} & \boxed{10} & \boxed{3} & \boxed{X} \\ \boxed{10} & \boxed{3} & \boxed{3} & \boxed{10} & \boxed{X} \\ \boxed{3} & \boxed{10} & \boxed{10} & \boxed{3} & \boxed{X} \\ \boxed{3} & \boxed{10} & \boxed{3} & \boxed{10} & \boxed{X} \\ \boxed{3} & \boxed{3} & \boxed{10} & \boxed{10} & \boxed{X} \end{array}\right\}$$ 6 ways $\times$ 5 places for X = Not 10 and Not 3

Thus there are 30 ways to do this:

This is also

$$M_5(2,2,1) = \frac{5!}{2!\,2!\,1!} = 30$$

$$\text{Prob} = \frac{30 \cdot 4 \cdot 3 \cdot 4 \cdot 3 \cdot (52-8)}{P_{52,5}} \cong 0.06\%$$

(3) I have 30 books. 5 are labeled classics, 10 are labeled mysteries, 7 are labeled science, and the rest are sports. If I randomly select 6 books, what is the probability I

a) select at least 2 science books?

Since order doesn't matter: $|\Omega| = \binom{30}{6} = \frac{30!}{6!\,24!} = 593,775$

$$\text{Prob} = \frac{\sum_{k=2}^{6} \binom{7}{k}\binom{23}{6-k}}{\binom{30}{6}}$$

b) only select books that are mysteries or sports books?

$$\text{Prob} = \frac{\sum_{k=0}^{6} \binom{10}{k}\binom{8}{6-k}}{\binom{30}{6}} = \frac{\binom{18}{6}}{\binom{30}{6}}$$

c) select books with precisely 2 being classics and 1 being science.

$$\text{Prob} = \frac{\binom{5}{2}\binom{7}{1}\binom{18}{3}}{\binom{30}{6}}$$

(4) Consider an experiment where you roll two fair, 4-sided dice. Label one as die 1 and one as die 2. Let X be the random variable which is the sum of the values on die 1 and die 2. Let Y be a random variable which is the value of die 1 minus the value of die 2.

D2 \ D1	1	2	3	4
1	2	3	4	5
2	3	4	5	6
3	4	5	6	7
4	5	6	7	8

a) Find the pmfs for X and Y individually. Write them as tables.

x	2	3	4	5	6	7	8
$f_X(x)$	1/16	3/16	3/16	4/16	3/16	2/16	1/16

D2 \ D1	1	2	3	4
1	0	1	2	3
2	-1	0	1	2
3	-2	-1	0	1
4	-3	-2	-1	0

y	-3	-2	-1	0	1	2	3
$f_Y(y)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

b) Find the joint pmf of X and Y. Write it as a table. Are X and Y independent? Explain.

Y \ X	2	3	4	5	6	7	8
-3	0	0	0	1/16	0	0	0
-2	0	0	1/16	0	1/16	0	0
-1	0	1/16	0	1/16	0	1/16	0
0	$\frac{1}{16}$	0	1/16	0	1/16	0	1/16
1	0	1/16	0	1/16	0	1/16	0
2	0	0	1/16	0	1/16	0	0
3	0	0	0	1/16	0	0	0

Not independent.

$$f_{X,Y}(2,-3) = 0$$

$$f_X(2) \cdot f_Y(-3) = \frac{1}{16} \cdot \frac{1}{16}$$

c) Find $E(XY)$.

$$E(XY) = \sum_{x,y} xy \cdot f_{X,Y}(x,y)$$

$$= \sum_x x\Big(-3 \cdot f_{X,Y}(x,-3) - 2 f_{X,Y}(x,-2) - 1 f_{X,Y}(x,-1) + 1 \cdot f_{X,Y}(x,1) + 2 f_{X,Y}(x,2) + 3 f_{X,Y}(x,3)\Big)$$

$$= 5(-3)\cdot\frac{1}{16} + 4(-2)\cdot\frac{1}{16} + 6(-2)\cdot\frac{1}{16} + 3(-1)\cdot\frac{1}{16} + 5(-1)\cdot\frac{1}{16} + 7(-1)\cdot\frac{1}{16}$$
$$+ 3(1)\cdot\frac{1}{16} + 5(1)\cdot\frac{1}{16} + 7(1)\cdot\frac{1}{16} + 4(2)\cdot\frac{1}{16} + 6\cdot(2)\cdot\frac{1}{16} + 5\cdot(3)\cdot\frac{1}{16}$$

$$= 0$$

(5) Let X and Y be independent, discrete random variables. Suppose that

$$f_X(k) = f_Y(k) = p(1-p)^k \quad \text{for all } k = 0, 1, 2, \cdots$$

for some $0 < p < 1$. Show that for any $n \geq 0$,

$$P(X = k \mid X + Y = n) = \frac{1}{n+1} \quad \text{for any } 0 \leq k \leq n.$$

First note that for any $n \geq 0$

$$P(X+Y=n) = \sum_{\substack{x,y:\\ x+y=n}} f_{X,Y}(x,n) = \sum_{\substack{x,y:\\ x+y=n}} f_X(x) \cdot f_Y(y)$$

$$= \sum_{x=0}^{n} f_X(x) \cdot f_Y(n-x)$$

$$= \sum_{x=0}^{n} p(1-p)^x \cdot p(1-p)^{n-x}$$

$$= p^2 \cdot (1-p)^n \cdot \sum_{x=0}^{n} 1 = (n+1)\, p^2 \cdot (1-p)^n$$

In this case

$$P(X=k \mid X+Y=n) = \frac{P(X=k \text{ and } X+Y=n)}{P(X+Y=n)}$$

$$= \frac{P(X=k \text{ and } Y=n-k)}{P(X+Y=n)}$$

$$= \frac{f_X(k) \cdot f_Y(n-k)}{P(X+Y=n)}$$

$$= \frac{p(1-p)^k\, p(1-p)^{n-k}}{(n+1)\, p^2 (1-p)^n} = \frac{1}{n+1}$$

(6) a) Let X be a continuous random variable with pdf

$$f_X(t) = \exp[-t - \exp(-t)] \quad \text{for all } t \in \mathbb{R}.$$

Find $F_X(x)$.

$$F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(t)\,dt = \int_{-\infty}^{x} \exp[-t - \exp(-t)]\,dt$$

$$\exp[-t-\exp(-t)] = e^{-t} \cdot e^{-e^{-t}}$$

$u = e^{-t}$

$du = -e^{-t}\,dt$

$$= -\int_{\infty}^{e^{-x}} e^{-u}\,du$$

$$= e^{-e^{-x}}$$

b) Find the real number a for which

$$f_X(x) = \begin{cases} a(x+1) & -1 \le x \le 0 \\ a(x-1)^2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

is the probability density function for a continuous random variable X.

If $f_X(x)$ is a pdf, then

$$1 = \int_{-\infty}^{\infty} f_X(x)\,dx = a\int_{-1}^{0} (x+1)\,dx + a\int_{0}^{1} (x-1)^2\,dx$$

$u = x+1$, $du = dx$ $\qquad$ $u = x-1$, $du = dx$

$$= a\int_0^1 u\,du + a\int_{-1}^{0} u^2\,du$$

$$= a \cdot \frac{u^2}{2}\Big|_0^1 + a\,\frac{u^3}{3}\Big|_{-1}^{0}$$

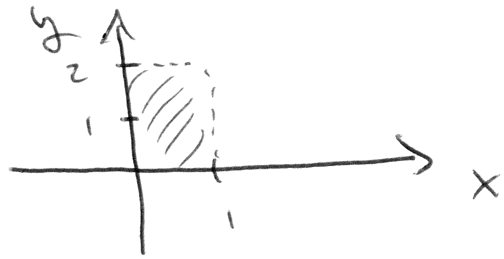
$$= \frac{a}{2} + \frac{a}{3} = \frac{5}{6}a$$

$$\Rightarrow \boxed{a = \frac{6}{5}}$$

7. Since X is uniformly distributed on $[0,1]$ and Y is uniformly distributed on $[0,2]$ the random variable $Z = X + Y$ clearly satisfies

$$0 \leq z \leq 3$$

Let's graph the domain of interest:



Since X and Y are independent, we also know that

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y) = \begin{cases} \frac{1}{2} & 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

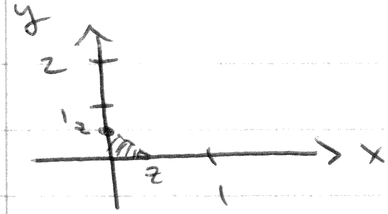
a) From the bounds above, it is clear that

$$F_Z(z) = \begin{cases} 0 & z \leq 0 \\ ?_0 & 0 \leq z \leq 3 \\ 1 & z \geq 3 \end{cases}$$

we need only find the cdf for $0 \leq z \leq 3$.

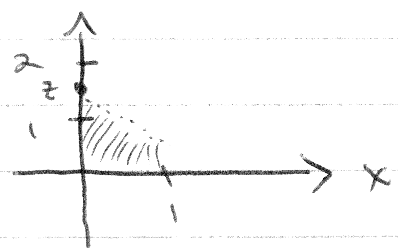
Case 1: Suppose $0 \le z \le 1$.

Then we want:



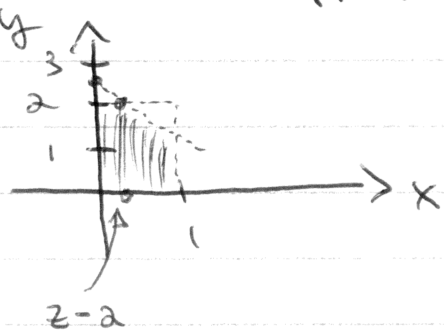
$$F_Z(z) = P(Z \le z)$$
$$= P(X + Y \le z)$$
$$= \int_0^z \int_0^{z-x} \tfrac{1}{2}\, dy\, dx$$
$$= \frac{1}{2} \int_0^z (z-x)\, dx$$
$$= \frac{z^2}{4}$$

Case 2: Suppose $1 \le z \le 2$



$$F_Z(z) = P(Z \le z)$$
$$= P(X + Y \le z)$$
$$= \int_0^1 \int_0^{z-x} \tfrac{1}{2}\, dy\, dx$$
$$= \frac{1}{2} \int_0^1 (z-x)\, dx$$
$$= \frac{1}{2}\left(z - \frac{1}{2}\right)$$

Case 3: Suppose $2 \le z \le 3$



$$F_Z(z) = P(Z \le z)$$
$$= P(X + Y \le z)$$
$$= \int_0^{z-2} \int_0^{2} \tfrac{1}{2}\, dy\, dx + \int_{z-2}^{1} \int_0^{z-x} \tfrac{1}{2}\, dy\, dx$$
$$= z - 2 + \frac{1}{2} \int^{1} (z-x)\, dx$$

For $2 \le z \le 3$

$$F_Z(z) = z-2 + \frac{1}{2}\int_{z-2}^{1} (z-x)\,dx$$

$$= z-2 + \frac{1}{2}\left(z\left(1-(z-2)\right) - \frac{x^2}{2}\Big|_{z-2}^{1} \right)$$

$$= z-2 + \frac{1}{2}z(3-z) - \frac{1}{4}\left(1-(z-2)^2\right)$$

$$= -\frac{1}{4}(z-5)(z-1)$$

So

$$F_Z(z) = \begin{cases} 0 & z \le 0 \\ z^2/4 & 0 \le z \le 1 \\ \frac{1}{2}\left(z-\frac{1}{2}\right) & 1 \le z \le 2 \\ -\frac{1}{4}(z-5)(z-1) & 2 \le z \le 3 \\ 1 & z \ge 3 \end{cases}$$

b)

$$f_Z(z) = \begin{cases} 0 & z \le 0 \\ \frac{z}{2} & 0 \le z \le 1 \\ \frac{1}{2} & 1 \le z \le 2 \\ -\frac{1}{2}(z-3) & 2 \le z \le 3 \\ 0 & z \ge 3 \end{cases}$$

c) We know that

$$E(Z) = E(X+Y) = E(X) + E(Y)$$
$$= \frac{1}{2} + 1$$
$$= \frac{3}{2}$$

Moreover

$$E(Z) = \int_{-\infty}^{\infty} z f_Z(z)\,dz = \int_0^1 z \cdot \frac{z}{2}\,dz + \int_1^2 z \cdot \frac{1}{2}\,dz$$
$$+ \int_2^3 -\frac{z}{2}(z-3)\,dz$$
$$= \frac{1}{2}\frac{z^3}{3}\Big|_0^1 + \frac{1}{2}\frac{z^2}{2}\Big|_1^2$$
$$-\frac{1}{2}\frac{z^3}{3}\Big|_2^3 + \frac{3}{2}\frac{z^2}{2}\Big|_2^3$$
$$= \frac{1}{6} + 1 - \frac{1}{4} - \frac{9}{2} + \frac{4}{3}$$
$$+ \frac{27}{4} - 3$$
$$= \frac{2}{12} + \frac{12}{12} - \frac{3}{12} - \frac{54}{12} + \frac{16}{12} + \frac{81}{12} - \frac{36}{12}$$
$$= \frac{18}{12} = \frac{3}{2}$$ ✓