Strong Subadditivity of the Von Neumann Entropy

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1 Subadditivity of Classical Shannon Entropy

The Shannon Entropy of a classical probability ditribution $p(a)$ over random variable a is defined as:

$$S[P(a)] = \sum_a -p(a)\,\log p(a) \tag{1}$$

In a sense, this formula measures how "unlocalized" the probability distribution is.

Let $P(x, y)$ be a probability distribution on two classical random variables x and y. Given this, we can talk about the probability of the value x without reference to y if we integrate P over all y values for a given x. Define this as P_x:

$$P_x(x) = \int P(x, y)dy, \tag{2}$$

and similarly for P_y. The statement of subadditivity is that $S[P(x, y)] \leq S[P(x)] + S[P(y)]$. A stronger statement can be made as well:

$$S[P(x, y, z)] + S[P(y)] \leq S[P(x, y)] + S[P(y, z)] \tag{3}$$

This statement can be rephrased using relative entropy: $H[P(x, y, z), P(x, y)P(y)^{-1}P(y, z)] \geq 0$. By the convexity of the function $f(x) = x \log x$, relative entropy is always greater than or equal to zero[1], being equal only when the two distributions are identical. So equality in the strong subadditivity holds only when

$$\frac{P(x, y, z)}{P(x, y)} = \frac{P(y, z)}{P(y)} \tag{4}$$

This is an equality among conditional probabilities:

$$P(z\,|x, y) = P(z\,|y) \tag{5}$$

2 Subadditivity of the Quantum Entropy

In the quantum picture, the analog of the positivity of relative entropy is Klein's matrix inequality:

Theorem 1. *Let A and B be nonnegative matrices with Ker(B) $\subseteq$ Ker(A). Then* $Tr[A(\log A - \log B])\geq Tr[A - B]$.

If A and B are density matrices, then the right-hand side vanishes. The proof of Klein's inequality comes from Talor's remainder theorem. See [3] for a proof of the general Klein's inequality. All subadditivity proofs presented here rely on this theorem, just like how classical subadditivity follows from positivity of relative entropy.

[1] needs proof

Theorem 2 (Subadditivity). $S[\rho_{12}] \leq S[\rho_1] + S[\rho_2]$

Proof. In Klein's inequality, let $A = \rho_{12}$ and $B = \rho_1 \otimes \rho_2$ where ρ_1 and ρ_2 are partial traces of the overall density matrix ρ_{12}. It is straightforward to check that $\rho_1 \otimes \rho_2 \mathbf{v} = \mathbf{0} \Rightarrow \rho_{12}\mathbf{v} = \mathbf{0}$. Klein's inequality now reads

$$Tr[\rho_{12} \log \rho_{12}] - Tr[\rho_{12} \log \rho_1 \otimes \rho_2] \geq Tr[\rho_{12} - \rho_1 \otimes \rho_2], \tag{6}$$

The right hand side of which vanishes since the trace of a difference is the difference of traces and density matrices have trace 1. The second term on the left can be written as

$$\begin{aligned} &Tr[\rho_{12}(\log \rho_1 \otimes \mathbb{1} + \mathbb{1} \otimes \log \rho_2] \\ &= Tr[\rho_1 \log \rho_1] + Tr[\rho_2 \log \rho_2] \\ &= -S[\rho_1] - S[\rho_2] \end{aligned} \tag{7}$$

where the first equality utilizes the linearity of the trace function and the addition property of log. So the result is

$$-S[\rho_{12}] + S[\rho_1] + S[\rho_2] \geq 0 \tag{8}$$

□

Ultimately, the goal of this review is to show that

$$S[\rho_{123}] + S[\rho_2] \leq S[\rho_{12}] + S[\rho_{23}]$$

Using Klein's inequality, it is possible to get closer to this goal if in addition we use the Golden-Thompson-Symanzik inequality which states that

Theorem 3 (Golden-Thompson-Syzmanzik). *$Tr[e^A e^B] \geq Tr[e^{A+B}]$ where A and B are self-adjoint matrices with equality holding if and only if A and B commute.*

The Golden-Thompson inequality is a consequence of the fact that $\mathrm{Tr}[(e^{A/2^k} e^{B/2^k})^{2^k}]$ is monotone decreasing in k and $\lim_{k\to\infty}(e^{A/2^k} e^{B/2^k})^{2^k} = e^{A+B}$. Now in Klein's inequality, let $A = \rho_{123}$, and let $\log B = \log \rho_{12} \otimes \mathbb{1}_{\mathcal{H}_3} + \mathbb{1}_{\mathcal{H}_1} \otimes \log \rho_{23}$:

$$\begin{aligned} &Tr[\rho_{123}(\log \rho_{123} - \log \rho_{12} \otimes \mathbb{1} - \mathbb{1} \otimes \log \rho_{23})] \\ =& - S[\rho_{123}] + S[\rho_{12}] + S[\rho_{23}] \\ \geq& \ 1 - Tr[e^{\log \rho_{12} \otimes \mathbb{1} + \mathbb{1} \otimes \log \rho 23}], \end{aligned} \tag{9}$$

where the first equality is for the same reason as that in (7). Applying the Golden-Thompson inequality to the right-hand side gives

$$\begin{aligned} -S[\rho_{123}] + S[\rho_{12}] + S[\rho_{23}] \geq& 1 - Tr[e^{\log \rho_{12} \otimes \mathbb{1}} e^{\mathbb{1} \otimes \log \rho 23}] \\ =& 1 - Tr_{123}[(\rho_{12} \otimes \mathbb{1})(\mathbb{1} \otimes \rho_{23})] \\ =& 1 - Tr_{23}[Tr_1[\rho_{12} \otimes \mathbb{1}]\rho_{23}] \\ =& 1 - Tr_2\,[\rho_2 Tr_3[\rho_{23}]] \\ =& 1 - Tr_2[\rho_2^2] \\ \geq& 1 - 1 = 0, \end{aligned} \tag{10}$$

where the inequality in the last line is due to the purity measure of density matrices being at most one. This proves that

$$S[\rho_{123}] \leq S[\rho_{12}] + S[\rho_{23}]. \tag{11}$$

The proof of strong subadditivity proceeds parallel to above: Use the left hand side of Klein's inequality to introducec the entropy of relevant subsystems and try to show the right hand side is greater than or equal to zero. For strong subadditivity the number of subsystems introduced is three, so we need an analog of The Golden-Thompson Inequality that involves the exponential of three matrices rather than two. Unfortunately, the natural extension of the Golden-Thompson statement $Tr[e^{A+B+C}] \leq Tr[e^A e^B e^C]$ does not hold true. But there is an inequality by Lieb that successfully relates three matrices:

Theorem 4 (Lieb's Triple Matrix Inequality)**.** *For all positive matrices R, S, and T,*

$$Tr[e^{\log R+\log S+\log T}] \leq Tr\left[\int_0^\infty R\frac{1}{S+uI}T\frac{1}{S+uI}du\right]. \tag{12}$$

Proof. Lieb's theorem follows from two lemmas which will be given without proof here.

Lemma 1. *The function $A \mapsto Tr[e^{K+\log A}]$ is concave in A for any Hermitian matrix K.*

Lemma 2. *For any function F concave in A, $F(B) \leq \lim_{x\to 0} \frac{F(A+xB)-F(B)}{x}$.*

In lemmas 1 and 2, let $K = \log R - \log S$ (assuming R & S are Hermitian), $A = S$, and $B = T$. With $F(A) = Tr[e^{K+\log A}]$, lemma 2 then reads

$$Tr[e^{\log R-\log S+\log T}] \leq \lim_{x\to 0}\frac{Tr[e^{\log R-\log S+\log(S+xT)}]-Tr[R]}{x}. \tag{13}$$

The integral expression in Lieb's theorem can be recovered by using a representation of matrix logarithms:

$$\log(S+xT)-\log S = \int_0^\infty \frac{1}{S+uI}xT\frac{1}{S+xT+uI}du. \tag{14}$$

Up to first order in x, the trace of the exponential in the numerator of 13 can be written

$$Tr[e^{\log R+x\int_0^\infty [1/(S+uI)]T[1/(S+xT+uI)]du}] = Tr[R]+xTr\left[R\int_0^\infty \frac{1}{S+uI}T\frac{1}{S+uI}du\right], \tag{15}$$

at which point (13) becomes Lieb's inequality. □

Theorem 5 (Strong Subadditivity)**.** $S[\rho_{123}] \leq S[\rho_{12}] + S[\rho_{23}] - S[\rho_2]$.

Proof. In Klein's inequality, let $A = \rho_{123}$ as usual, but let $\log B = \log\rho_{12}\otimes\mathbb{1}_{\mathcal{H}_3}+\mathbb{1}_{\mathcal{H}_1}\otimes\rho_{23}-\mathbb{1}_{\mathcal{H}_1}\otimes\rho_2\otimes\mathbb{1}_{\mathcal{H}_3}$:

$$\begin{aligned}
-S[\rho_{123}+S[\rho_{12}]+S[\rho_{23}]-S[\rho_2] \geq& Tr[\rho_{123}]-Tr[e^{\log\rho_{12}\otimes\mathbb{1}+\mathbb{1}\otimes\log\rho_{23}-\mathbb{1}\otimes\rho_2\otimes\mathbb{1}}]\\
\geq& 1-Tr\left[\int_0^\infty(\rho_{12}\otimes\mathbb{1})\frac{1}{\mathbb{1}\otimes\rho_2\otimes\mathbb{1}+uI}(\mathbb{1}\otimes\rho_{23})\frac{1}{\mathbb{1}\otimes\rho_2\otimes\mathbb{1}+uI}du\right] \text{, by Lieb}\\
=& 1-Tr_2\left[\rho_2\frac{1}{\rho_2+uI}\rho_2\frac{1}{\rho_2+uI}du\right]\\
=& 1-Tr_2[\rho_2]\\
=& 0,
\end{aligned} \tag{16}$$

which concludes the proof. □

The equality condition for theorem 5 is that for Klein's inequality:

$$\rho_{123}-\rho_{12} = \rho_{12}-\rho_1$$

. Notice the parallelity with the classical conditions for equality.

3 Discussion

It is important to stress that the strong subadditivity of entropy does not follow from subadditivity. In the case where the system consists of three subsystems and the density matrix lives in a tensor product state of three Hilbert spaces, subadditivity can at most state that

$$S[\rho_{123}] \leq S[\rho_1] + S[\rho_2] + S[\rho_3], \tag{17}$$

which is a weaker statement than theorem 5. Subadditivty neglects all correlations among subsystems and says that the total information of the individual subsystems exceeds that of the full system. Strong subadditivity says that you exceed the information content of the full system even if you add up the subentropies neglecting only some of the correlations.

It is tempting to make the claim that strong subadditivity specifically neglects the correlations that involve all three subsystems but not those between pairs of subsytems. Mathematically speaking, this statement would be written as:

$$S[\rho_{123}] \leq S[\rho_{12}] + S[\rho_{23}] + S[\rho_{13}] - S[\rho_1] - S[\rho_2] - S[\rho_3], \tag{18}$$

which unfortunately does not follow from strong subadditivity, since it can be obtained from theorem 5 by adding $S[\rho_{13}] - S[\rho_1] - S[\rho_3]$ to the right hand side. But by subadditivity, this is a negative quantity, so the statement of (18) cannot be made with the results developed in this review. Further work is needed to prove or disprove this even stronger subbaditivity.

References

[1] Mary Beth Ruksai *Inequalities for Quantum Etropy: A review with Consitions for equality* 2002: Journal of Mathematical Physics.

[2] Zoltán Kolarovski *Monotonicity of relative entropy, olden-Thompson inequality, Lieb's triple matrix inequality & strong subadditivity of quantum entropy* 2018.

[3] Eric Carlen *Trace Inequalities and Quantum Entropy* 2009.