On $C^*$-algebras

In previous classes, we often discussed the collection of "observables" of a quantum system as the "algebra of observables" for a given quantum system. This phrase has meaning, and today we discuss this in detail.

Let us begin by observing important properties in one of the most important examples.

On $B(\mathcal{H})$:

Let $\mathcal{H}$ be a complex Hilbert space. By $B(\mathcal{H})$ we denote the collection of all bounded linear operators on $\mathcal{H}$. Recall that when $\mathcal{H}$ is finite dimensional, then $B(\mathcal{H})$ is just a collection of matrices.

- It is clear that $B(\mathcal{H})$ is a vector space.
- Moreover, $B(\mathcal{H})$ is a normed vector space when it is equipped with

$$
\|A\| = \sup_{\|\psi\|=1} \frac{\|A\psi\|}{\|\psi\|} \quad \text{for all } A \in B(\mathcal{H}).
$$

The quantity above is called the operator norm and $\|A\|$ is said to be the norm of $A$. 
In terms of this norm, $B(H)$ is also a metric space. In fact,

$$d(A, B) = \| A - B \|$$

for all $A, B \in B(H)$ defines a metric on $B(H)$.

One readily checks that $B(H)$ is complete (as a metric space) with respect to this metric induced by the operator norm, and hence, $B(H)$ is a **Banach space**.

There is also a well-defined product on $B(H)$, it is just the composition of linear maps, and this product satisfies the following norm estimate:

$$\| AB \| \leq \| A \| \cdot \| B \|$$

for all $A, B \in B(H)$.

The above estimate is sufficient to imply that this product, i.e., the map $(A, B) \mapsto AB$, is norm continuous.

These facts together imply that $B(H)$ is a **Banach algebra**.
It will also be important that there is a well-defined map \( * : B(H) \to B(H) \) which satisfies certain properties.

On \( B(H) \), the * map is the map \( A \mapsto A^* \) the adjoint of \( A \in B(H) \). Recall that the adjoint map satisfies the following properties:

- \( (\alpha A + \beta B)^* = \overline{\alpha} A^* + \overline{\beta} B^* \) for all \( A, B \in B(H) \) and \( \alpha, \beta \in \mathbb{C} \).

(In words, the *-map is anti-linear.)

- \( (A^*)^* = A \) for all \( A \in B(H) \).

(In words, the *-map is an involution.)

One also has that

- \( (AB)^* = B^* A^* \) for all \( A, B \in B(H) \).

- \( \| A^* \| = \| A \| \) for all \( A \in B(H) \).

- \( \| A^* A \| = \| A \|^2 \) for all \( A \in B(H) \).

We will now declare that these properties we have seen in \( B(H) \) are useful and important as a structure. In the language of mathematics, we will say that anything that satisfies all these properties is a \( \text{C}^\star \)-algebra. This structure will play an important role later, when we investigate thermodynamic limits.
**Definition 1** A complex vector space $\mathcal{A}$ is said to be an algebra if it is equipped with a product map, i.e. $(\cdot, \cdot) : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, $(A, B) \mapsto AB$, for which

i) $A(BC) = (AB)C$ \hspace{1cm} \text{for all} \ A, B, C \in \mathcal{A}$

ii) $A(B+C) = AB + AC$ \hspace{1cm} \text{for all} \ A, B, C \in \mathcal{A}$

iii) $\alpha A(B) = (\alpha A)(B)$ \hspace{1cm} \text{for all} \ A, B \in \mathcal{A} \text{ and } \alpha \in \mathbb{C}$.

In words, a vector space with a product is an algebra. Often the above is referred to as an associative algebra, but we will just say algebra.

Let $\mathcal{A}$ be an algebra. An element $1 \in \mathcal{A}$ is said to be an identity element if

$$1A = A = A1 \hspace{1cm} \text{for all} \ A \in \mathcal{A}.$$  

Any algebra $\mathcal{A}$ with an identity element is called a unital algebra.

A subspace $\mathcal{B} \subseteq \mathcal{A}$ that is also an algebra with respect to the operations in $\mathcal{A}$ is called a subalgebra of $\mathcal{A}$. 
Definition: An algebra \( A \) is called a \(*\)-algebra if there is a map \( \ast: A \to A \), \( A \to A^* \), for which

i) \( (A^*)^* = A \) for all \( A \in A \) (i.e., \( \ast \) is an involution)

ii) \( (A B)^* = B^* A^* \) for all \( A, B \in A \) (i.e., \( \ast \) is an anti-morphism)

iii) \( (\alpha A + \beta B)^* = \bar{\alpha} A^* + \bar{\beta} B^* \) for all \( A, B \in A \) and \( \alpha, \beta \in \mathbb{C} \) (i.e., \( \ast \) is anti-linear)

Let \( A \) be a \(*\)-algebra. A subset \( B \subset A \) is said to be self-adjoint if \( A \in B \Rightarrow A^* \in B \).

Note: Self-adjoint subspaces are subspaces that are closed under the \(*\) map.

An element \( A \in A \) is said to be self-adjoint if \( A^* = A \).

If \( \text{Self}(A) \) denotes the collection of all self-adjoint elements of \( A \), it is clear that \( \text{Self}(A) \) is a self-adjoint subset of \( A \).
Definition 3. An algebra \( A \) is called a \underline{normed algebra} if there is a map \( \| \cdot \| : A \to \mathbb{R} \) for which

i) \( \| A \| \geq 0 \) for all \( A \in A \) and \( \| A \| = 0 \Leftrightarrow A = 0 \).

ii) \( \| xA \| = \| x \| \cdot \| A \| \) for all \( A \in A \) and \( x \in \mathbb{C} \).

iii) \( \| A + B \| \leq \| A \| + \| B \| \) for all \( A, B \in A \).

iv) \( \| AB \| \leq \| A \| \cdot \| B \| \) for all \( A, B \in A \).

Note: Properties i), ii), and iii) above show that a normed algebra is an algebra with a norm on it.

Property iv) shows that the norm "respects products" in the sense of this inequality.

Let \( A \) be a normed algebra. For each \( A \in A \), we call \( \| A \| \) the norm of \( A \). Since the norm on \( A \) is a norm, it defines a metric as follows:

\[ d(A, B) = \| A - B \| \quad \text{for all} \quad A, B \in A. \]

This is said to be the metric induced by this norm.

This metric can be used to define open balls (and hence open sets). This collection of open sets is a topology.

It is called the \underline{uniform or norm topology} on \( A \).
Let $\mathcal{A}$ be a normed algebra. If $\mathcal{A}$ is complete (as a metric space) with respect to the metric induced by its norm, then $\mathcal{A}$ is called a Banach algebra.

- A *-algebra $\mathcal{A}$ is said to be a normed *-algebra if $\mathcal{A}$ is a normed algebra and the norm satisfies

$$\|A^*\| = \|A\|$$

for all $A \in \mathcal{A}$.

(The above equality shows that the map $A \mapsto A^*$ is continuous in norm.)

- A normed *-algebra is said to be a Banach *-algebra if it is complete with respect to the metric induced by its norm.

We now present the main definition of the day.

**Definition** We say that a Banach *-algebra $\mathcal{A}$ is a C*-algebra if

$$\|A^*A\| = \|A\|^2$$

for all $A \in \mathcal{A}$.

The final equality above is often called the C*-property.
Example 1. Let $\mathcal{H}$ be a complex Hilbert space. Let $A = B(\mathcal{H})$, the collection of bounded linear operators over $\mathcal{H}$, with $\|\cdot\|$ being the operator norm and $*$ being the adjoint operation. $A = B(\mathcal{H})$ is a $C^*$-algebra.

We verified all the relevant properties before.

Note: $\mathcal{H}$ does not need to be finite dimensional.

Example 2. Let $A = B(C(\mathbb{R}^n))$, i.e., the collection of all functions $f : \mathbb{R}^n \to \mathbb{C}$ for which $f$ is bounded and continuous. This collection is clearly a vector space with a natural product. Let $*$ be complex conjugation and denote by $\|\cdot\|$ the supremum norm, i.e.,

$$\|f\| = \sup_{x \in \mathbb{R}^n} |f(x)|.$$

In this case, $A = B(C(\mathbb{R}^n))$ is a $C^*$-algebra.

Note: Since $fg = gf$ for all $f, g \in A$, $A$ is said to be a commutative $C^*$-algebra.

This is not the case with $B(H)$ if $\dim(\mathcal{H}) \geq 2$. 
Example 3 Let $H$ be a complex Hilbert space. Let $C \subseteq B(H)$ be the collection of compact operators on $H$. Then $C$ is a $C^*$-subalgebra of $B(H)$.

Note: If $\dim(H) = +\infty$, then $1 \notin C$.

Note: Here is a general fact:

Let $H$ be a complex Hilbert space. Let $A \subseteq B(H)$ be any norm closed subalgebra of $B(H)$ which is also a self-adjoint subset of $B(H)$. Then $A$ is a $C^*$ sub-algebra of $B(H)$.

It is not hard to prove this fact.

It is not hard to show that the compact operators satisfy these constraints.

Here is a useful fact.

Let $A \subseteq B(H)$ be a unital $C^*$-algebra.

Then $1 \in A$ is unique.

- $1^* = 1$
- $\|1\| = 1$. 

On positive elements

Let $\mathcal{A}$ be a C*-algebra. An element $A \in \mathcal{A}$ is said to be positive, which we denote by $A \geq 0$, if there is some $B \in \mathcal{A}$ and $A = B^* B$.

Note: We should probably say $A$ is non-negative.

Note: It is easy to check that: $A \geq 0 \Rightarrow A \in \mathcal{A}_{sa}$.

It is an important fact that this notion of positivity allows us to define a partial order on $\mathcal{A}_{sa}$.

Let $A, B \in \mathcal{A}_{sa}$. We write that $A \geq B$ if and only if $A - B \geq 0$. (Note: we may also write $B \leq A$.)

The following proposition contains some useful facts:

Proposition: Let $\mathcal{A}$ be a unital C*-algebra.

i) If $A \geq 0$ and $A \leq 0$, then $A = 0$.

ii) If $A \geq B$ and $B \geq C$, then $A \geq C$.

iii) If $A \geq 0$, then $\|A\| I \geq A$.

iv) If $A \geq B \geq 0$, then $C^* A C \geq C^* B C \geq 0$ for all $C \in \mathcal{A}$.