Lieb-Robinson Bounds

Recall: let \((\mathcal{P}, d)\) be a regular metric space and let \(F\) be an \(F\)-function on \((\mathcal{P}, d)\). We say that an interaction \(\Phi \in B_F(\mathcal{P})\) if

\[
\|\Phi\|_F = \sup_{x \neq y} \sum_{x \in \mathcal{P} : \Phi(x) \neq 0} \frac{\|\Phi(x)\|}{F(d(x, y))} < \infty.
\]

Said differently, if \(\Phi \in B_F(\mathcal{P})\), then for each pair \(x, y \in \mathcal{P}\)

\[
\sum_{x \in \mathcal{P} : \Phi(x) \neq 0} \|\Phi(x)\| \leq \|\Phi\|_F \cdot F(d(x, y)).
\]

**Theorem (Lieb-Robinson Bound)** Let \((\mathcal{P}, d)\) be a regular metric space and \(\Phi \in B_F(\mathcal{P})\) for some \(F\)-function \(F\) on \((\mathcal{P}, d)\). Let \(\mathcal{X}, \mathcal{X}'\) be finite disjoint sets. Let \(\mathcal{A} \subseteq \mathcal{P}\) be finite and satisfy \(\mathcal{X} \cup \mathcal{X}' \subseteq \mathcal{A}\).

Then, for any \(A \in \mathcal{A}_\mathcal{X}\) and \(B \in \mathcal{A}_\mathcal{X}'\), the bound

\[
\|\sum_{t \in \mathbb{T}} \gamma_t(A, B)\| \leq \frac{2 \|\Phi\|_F \|\Phi\|}{c_F} \left( e^{2 \frac{\|\Phi\|_F \|\Phi\|}{c_F}} - 1 \right) \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{X}'} F(d(x, y))
\]

holds for all \(t \in \mathbb{R}\).
Recall also

Last class we proved a norm-preservation lemma.

Lemma (Norm-Preservation). Let \( I \subseteq \mathbb{R} \) be an open interval and \( H \) be a complex Hilbert space. Let \( A, B : I \to B(H) \) satisfy

i) \( A(t)^* = A(t) \) for all \( t \in I \).

ii) \( t \mapsto A(t) \) and \( t \mapsto B(t) \) are norm continuous.

Then, the unique solution of the \( B(H) \)-valued initial value problem:

\[
\frac{d}{dt} y(t) = -i [A(t), y(t)] + B(t) \quad \text{for } t \in I \quad \text{with } y(t_0) = y_0 \in B(H)
\]

for some \( t_0 \in I \)

Satisfies the norm bound

\[
\| y(t) \| \leq \| y_0 \| + \int_{t_0}^t \| B(s) \| \, ds \quad \text{for all } t \in I.
\]

Lastly, the following identity will be useful.

Let \( H \) be a Hilbert space and \( A, B, C \in B(H) \). Then

\[
[ [A, B], C ] + [ [B, C], A ] + [ [C, A], B ] = 0
\]

This is called the Jacobi Identity. It is easy to check by just expanding the commutators. In words, if you sum this iterated commutator or cyclic permutations of \( (A, B, C) \), then you get 0.
Proof of the Lieb-Robinson Bound

Fix finite disjoint sets $\mathcal{X}, \mathcal{Y} \subset \mathcal{N}$. Let $\Lambda \subset \mathcal{N}$ be finite with $\mathcal{X} \cup \mathcal{Y} \subset \Lambda \setminus \mathcal{Y}$ and take $A \in \mathcal{A}_\mathcal{X}$ and $B \in \mathcal{A}_\mathcal{Y}$.

Define a function $f: \mathbb{R} \to B(\mathcal{H}_\Lambda) = \mathcal{A}_\Lambda$ by setting

\[ f(t) = \left[ \tau^t_{A \to B} (A), B \right] . \]

Our goal is to estimate the norm of this function. To do so, let's first calculate its derivative.

\[ f'(t) = \sum_{Z \subset \Lambda: Z \cap \Lambda \neq \emptyset} \left[ \tau^t_{A \to B} (A(Z)), B \right] \]

\[ = i \left[ \tau^t_{A \to B} \left( \sum_{Z \subset \Lambda: Z \cap \Lambda \neq \emptyset} d(Z), A \right) \right], B \left] \right. \]

\[ = i \sum_{Z \subset \Lambda: Z \cap \Lambda \neq \emptyset} \left[ \tau^t_{A \to B} (A(Z)), \tau^t_{A \to B} (A) \right], B \left] \right. \]

Now apply Jacobi with $A = \tau^t_{A \to B} (A(Z))$, $B = \tau^t_{A \to B} (A)$, and $E = B$

\[ \Rightarrow f'(t) = -i \sum_{Z \subset \Lambda: Z \cap \Lambda \neq \emptyset} \left[ \tau^t_{A \to B} (A(Z)), B \right], \tau^t_{A \to B} (A) \left] \right. \]

\[ - i \sum_{Z \subset \Lambda: Z \cap \Lambda \neq \emptyset} \left[ E \cap B, \tau^t_{A \to B} (A(Z)), \tau^t_{A \to B} (A) \right] \]
This can be re-written as:

\[ f'(t) = -i \left[ A(t), f(t) \right] + B(t) \]

with

\[ A(t) = -\sum_{z \in \Lambda; \ z \notin \Phi} \eta_t^{\Phi}(f(z)) \]

and

\[ B(t) = -i \sum_{z \in \Lambda; \ z \notin \Phi} \left[ \eta_t^{\Phi}(f), \eta_t^{\Phi}(f(z)), \beta \right] \]

One readily checks that:

i) \( A(t)^* = A(t) \) for all \( t \in \mathbb{R} \)

   and moreover

ii) \( t \to A(t) \) and \( t \to B(t) \) are both norm continuous.

In this case, the norm preservation lemma applies. Take \( t_0 = 0 \) and note that

\[ \|f(t)\| \leq \|f(0)\| + \sum_{s \in \mathbb{R}} \|f(s)\| \] 

holds for all \( t > 0 \).

A similar bound holds for \( t < 0 \)...

Note further that:

\[ \|B(t)\| \leq \sum_{z \in \Lambda; \ z \notin \Phi} \left\| \left[ \eta_t^{\Phi}(f), \eta_t^{\Phi}(f(z)), \beta \right] \right\| \leq 2 \|A\| \sum_{z \in \Lambda; \ z \notin \Phi} \|\eta_t^{\Phi}(f(z))\| \]
In this case, the norm-preservation lemma gives us the first estimate:

\[ \| E T_{f_0}^t (A) , B \| \leq \| E A, B \| + 2 \| A \| \sum_{zd: B \neq 0} \int_0^t \| E T_{f_0}^s (A(z)) , B \| ds \]

Note that the left-hand-side and the right-hand-side look similar. We would like to iterate this estimate.

To do so, it is convenient to define a quantity which focuses on support of an observable and time. Define

\[ C_B^t (z ; t) = \sup_{A \in Q_1: A \neq 0} \frac{\| E T_{f_0}^t (A) , B \|}{\| A \|} \]

Note that

\[ C_B^t (z ; t) \leq 2 \| B \| \] for all \( z \in A \) and \( t \in \mathbb{R} \)

Moreover

\[ C_B^t (z ; 0) \leq 2 \| B \| \sum_{\frac{z}{2\pi} \neq \phi} \] for all \( z \in A \) and \( t \in \mathbb{R} \)

Where we have set

\[ \delta_{\frac{z}{2\pi}} (z) = \begin{cases} 1 & \text{if } \frac{z}{2\pi} \neq \phi \\ 0 & \text{otherwise} \end{cases} \]
Rewriting the first estimate in terms of this new quantity yields

\[ C_{B}^{\Lambda}(x, t) \leq C_{B}(x, 0) + 2 \sum_{z \in \Lambda_{1}} \left( \sum_{s \geq 0} C_{B}(z, s) ds \right) \]

\[ \leq 2 \| \mathbf{b} \|_{\infty} \delta_{\tau}(x) + 2 \sum_{z \in \Lambda_{1}} \left( \sum_{s \geq 0} C_{B}(z, s) ds \right) \]

Now iterate:

\[ \leq 2 \| \mathbf{b} \|_{\infty} \delta_{\tau}(x) + 2 \sum_{z \in \Lambda_{1}} \left( \sum_{s \geq 0} C_{B}(z, s) ds \right) \]

\[ + 2 \sum_{z \in \Lambda_{1}} \left( \sum_{s \geq 0} C_{B}(z, s) ds \right) \]

\[ + 2 \sum_{z \in \Lambda_{1}} \left( \sum_{s \geq 0} C_{B}(z, s) ds \right) \]

\[ \ldots \text{iterating } N \geq 3 \text{ times produces:} \]

\[ C_{B}^{\Lambda}(x, t) \leq 2 \| \mathbf{b} \|_{\infty} \delta_{\tau}(x) + \sum_{n=1}^{N} \mathop{\text{an}} \left( \frac{(2t)^{n}}{n!} \right) \] + \text{remainder term}

Keep track of 2 and integration over simplex.
For each \( n \geq 1 \), we have found that:

\[
q_n = \sum_{z_1 \in \mathbb{C}} \sum_{z_2 \in \mathbb{C}} \cdots \sum_{z_{n+1} \in \mathbb{C}} \left( \frac{N+1}{11\delta(2z_1)} \right) S_{n+1}^{\infty}(z_n)
\]

and moreover, the remainder term is given by

\[
R_{n+1}(t) = 2^{N+1} \sum_{z_1 \in \mathbb{C}} \sum_{z_2 \in \mathbb{C}} \cdots \sum_{z_{N+1} \in \mathbb{C}} \left( \frac{N+1}{11\delta(2z_1)} \right) \times
\]

\[
\times \sum_{s_1} \ldots \sum_{s_N} c_B^r(z_{s_1}, s_{s_2}, \ldots, s_N) ds_{s_1} ds_{s_2} \ldots ds_N
\]

To complete the proof, we first demonstrate that:

\[ R_{n+1}(t) \to 0 \text{ as } n \to \infty. \]

This will be true uniformly for \( t \) in compact subsets of \( \mathbb{R} \). Once this is proven, we have established that:

\[
C_B(x, t) \leq \sup_{x \in \mathbb{R}} \left[ S_{n+1}^\infty(x) + \sum_{n=1}^{\infty} \frac{q_n (2x)^n}{n!} \right] \text{ for all } t \geq 0.
\]

The claimed bound will follow from an estimate of \( q_n \) for all \( n \geq 1 \). The estimate for \( R_{n+1}(t) \) and \( q_n \) are similar.

We start with the remainder.
Recall the apriori estimate:

\[ (b_{(2,t)} \leq 2^{11}B_{11} \quad \text{for all } z \in \mathbb{C} \text{ and } t \in \mathbb{R}. \]

In this case, for any \( N \geq 1 \)

\[
\begin{align*}
&\sum_{t} \sum_{s} \cdots \sum_{u} (\sum_{n=1}^{N+m} (s_1, s_2, \ldots, s_u) ds_1 \cdots ds_u) \\
&\leq 2^{11}B_{11} \cdot \sum_{t} \sum_{s} \cdots \sum_{u} ds_1 \cdots ds_u \\
&= 2^{11}B_{11} \left( \frac{(N+1)^{N+1}}{(N+1)!} \right) = 2^{11}B_{11} \cdot \frac{t^{N+1}}{(N+1)!}. \\
\end{align*}
\]

Thus the remainder can be estimated by

\[
R_{N+1}(t) \leq 2^{11}B_{11} \cdot \frac{(2t)^N}{(N+1)!} \cdot \sum_{z_1 \in \mathbb{C}} \sum_{z_2 \in \mathbb{C}} \sum_{z_3 \in \mathbb{C}} \cdots \sum_{z_{N+1} \in \mathbb{C}} \frac{N+1}{114(2^{N+1})}.
\]

To estimate this sum, we 1st recognize it as a sum over "chains of sets" which emanate from \( x \).
One can visualize this sum as follows:

The fact that the sets intersect means that:

There is some:  
\( w_1 \in Z_1 \setminus N \)
\( w_2 \in Z_2 \setminus N \)
\( w_3 \in Z_3 \setminus N \)
\( \vdots \)
\( w_N \in Z_N \setminus N \)

For convenience, we also include a notation for some

\( w_{N+1} \in N \setminus N \)

By our counting sets, it is clear that:

\[ \sum \sum \cdots \sum \neq \sum \]
\( \subset_1 \subset \subset_2 \subset \subset_N \subset \subset_N \)

\[ \sum \sum \cdots \sum \sum \]
\( \subset_1 \subset \subset_2 \subset \subset_N \subset \subset_N \)

for any non-negative quantity \( \ast \).
For the remainder, the sum of interest is:

\[
\sum_{j=1}^{\infty} \sum_{z_{n+1} \in \mathbb{Z}^d} \left( \frac{1}{\gamma} \| \psi(z_{j+n}) \| \right)
\]

\[
\leq \sum_{w, \bar{w} \in \mathbb{Z}^d} \sum_{k=1}^{\infty} \sum_{z_{n+1} \in \mathbb{Z}^d} \left( \frac{1}{\gamma} \| \psi(z_{j+n}) \| \right)
\]

Note that the final sum is:

\[
\sum_{z_{n+1} \in \mathbb{Z}^d} \| \psi(z_{n+1}) \| \leq \| \alpha \|_E \cdot F(d(w_{n+1}, w_{n+2}))
\]

In fact, this is true for all:

\[
\sum_{z \in \mathbb{Z}^d} \| \psi(z) \| \leq \| \alpha \|_E \cdot F(d(w_{k}, w_{k+1}))
\]

\[\Rightarrow R_{n+1}(\alpha) \leq 2 \cdot \| \alpha \|_E \left( \frac{2 \alpha}{(n+1)!} \right) \sum_{j=1}^{\infty} \sum_{z_{n+1} \in \mathbb{Z}^d} \left( \frac{1}{\gamma} \| \psi(z_{j+n}) \| \right)
\]

\[
\leq 2 \cdot \| \alpha \|_E \left( \frac{2 \alpha}{(n+1)!} \right) \sum_{\mathbf{w} \in \mathbb{Z}^d} \sum_{k=1}^{\infty} \sum_{z_{n+1} \in \mathbb{Z}^d} \left( \frac{1}{\gamma} \| \psi(z_{j+n}) \| \right) F(d(w_{k}, w_{k+1}))
\]
Now we use the property of the $F$ function.

Note that

$$
\sum_{w_{n+1} \in \Lambda} F(d(w_n, w_{n+1})). F(d(w_{n+1}, w_{n+2})) \leq C_F \cdot F(d(w_n, w_{n+2}))
$$

We can apply this for all $k$ with $2 \leq k \leq N+1$. We find that

$$
R_{N+1}(4) \leq 2.11 \beta_1 \cdot \sum_{w, \epsilon \in \Lambda} \sum_{w_{n+2} \in \Lambda} F(d(w, w_{n+2}))
$$

Sum this out

$$
\leq 2.11 \|B\| \cdot \|F\| \cdot \sum_{w, \epsilon \in \Lambda} \sum_{w_{n+2} \in \Lambda} 1
$$

$$
\leq 2.11 \beta_1 \cdot \|B\| \cdot \|F\| \cdot 18 \leq \text{the cardinality of } \Lambda
$$

Thus $R_{n+1}(4) \to 0$ as $n \to \infty$ and this is uniform for $t$ in compact subsets of $\mathbb{T}_2$.

We need only estimate the remaining series.
Since we have proven that the remainder term goes to zero, we have proven that
\[ C_B(\mathbf{z}; t) \leq 2 \| \mathbf{b} \| \left[ \delta_1(\mathbf{x}) + \sum_{n=1}^{\infty} \frac{c_n}{n^s} \right] \text{ for all } t \geq 0. \]

Let us now estimate the coefficients \( c_n \) for \( n \geq 1 \).

**Note:**

\[ q_1 = \sum_{z, c_1} \sum_{z, \mathbf{z} \neq \phi} \sum_{x \in \mathbf{z}} \sum_{y \in \mathbf{z}} \sum_{\mathbf{z}} \left[ \| d(x, y) \| \right] \]

\[ \leq \sum_{x \in \mathbf{z}} \sum_{y \in \mathbf{z}} \sum_{\mathbf{z}} \left[ \| d(x, y) \| \right] \]

\[ \leq \| \mathbf{d} \| \sum_{x \in \mathbf{z}} \sum_{y \in \mathbf{z}} F(d(x, y)) \]

Similarly,

\[ q_2 = \sum_{z, c_1} \sum_{z, \mathbf{z} \neq \phi} \sum_{x \in \mathbf{z}} \sum_{y \in \mathbf{z}} \sum_{\mathbf{z}} \left[ \| d(x, y) \| \right] \]

\[ \leq \sum_{x \in \mathbf{z}} \sum_{y \in \mathbf{z}} \sum_{\mathbf{z}} \left[ \| d(x, y) \| \right] \]

\[ \leq \| \mathbf{d} \| \sum_{x \in \mathbf{z}} \sum_{y \in \mathbf{z}} \sum_{\mathbf{z}} \left[ \| d(x, y) \| \right] \]

\[ \leq C = \| \mathbf{d} \| \sum_{x \in \mathbf{z}} \sum_{y \in \mathbf{z}} \sum_{\mathbf{z}} \left[ \| d(x, y) \| \right] \]
Now, \( n \geq \lambda \) we have that

\[
\begin{align*}
q_n &= \sum_{z \in \mathcal{C}} \left( \sum_{z_{nc} \in \mathcal{C}} \left( \sum_{j=1}^{\lambda} \left( \prod_{k=1}^{\lambda} \ell_{j_k}(y_{j_k}) \right) \right) \right) \left( \sum_{2n \in \mathcal{C}} \right) (2^n) \\
&\leq \sum_{x \in \mathcal{E}} \sum_{y \in \mathcal{E}} \sum_{w \in \mathcal{E}} \sum_{v \in \mathcal{E}} \sum_{u \in \mathcal{E}} \sum_{t_c \in \mathcal{C}} \left( \sum_{2^n \in \mathcal{C}} \left( \prod_{j=1}^{\lambda} \ell_{j}(y_{j}) \right) \right) F(d(x,y)) \\
&\leq (\lambda+1)^n \sum_{x \in \mathcal{E}} \sum_{y \in \mathcal{E}} F(d(x,y))
\end{align*}
\]

Thus

\[
\begin{align*}
C_{\beta}(\varepsilon, t) &= 2ll \left[ \sum_{x} S_{\beta}(x) \right] + \sum_{n=1}^{\infty} q_n (2t)^n \\
&\leq 2ll \left[ \sum_{x} S_{\beta}(x) \right] + \sum_{n=1}^{\infty} (\lambda+1)^n \sum_{x \in \mathcal{E}} \sum_{y \in \mathcal{E}} F(d(x,y)) (2t)^n \\
&= 2ll \left[ \sum_{x \in \mathcal{E}} \left( \frac{2\pi}{\lambda+1} \right)^t - 1 \right] \sum_{x \in \mathcal{E}} \sum_{y \in \mathcal{E}} F(d(x,y)).
\end{align*}
\]

This proves the theorem for \( t > 0 \). \( t < 0 \) is similar.