On Tensor Products

Any two quantum systems, described by Hilbert spaces $H_1$ and $H_2$, can be considered as a composite system. The Hilbert space for the composite system is given by the tensor product of $H_1$ and $H_2$. This notion readily extends to any finite collection of Hilbert spaces.

Goals for today:

1. Characterize the tensor product of $H_1$ and $H_2$.
2. Realize this tensor product in some important examples.

Some background in linear algebra:

A metric space, in particular a Hilbert space, is said to be separable if there is a countable dense subset. For this lecture, we will only consider separable, complex Hilbert spaces.

Note:

- All finite dimensional Hilbert spaces are separable.
- If $(X, \mu)$ is a finite measure space, then $L^2(X, d\mu)$ is separable.

In particular, $L^2(\mathbb{R}^n, dx)$ is separable for any $n \geq 1$. 
Fact: Every separable Hilbert space has a countable orthonormal basis (ONB).

- when $H$ is finite dimensional, this statement is clear.
- when $H$ is infinite dimensional, the notion of an ONB will be reviewed in the notes online.

Briefly: a sequence $\{x_n\}_{n=1}^{\infty}$ in $H$ is said to be an ONB if

i) $\langle x_n, x_m \rangle = \delta_{n,m}$ for all $n, m \geq 1$ (i.e., the sequence is an orthonormal set.)

and

ii) for every $x \in H$,

$$x = \lim_{N \to \infty} \sum_{n=1}^{N} \langle x, x_n \rangle x_n \quad (= \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n)$$

Here the limit is in norm.

Two consequences of this are:

- **(Completeness)** If $x \in H$ and $\langle x, x_n \rangle = 0$ for all $n \geq 1$, then $x = 0$.

- **(Parseval)** For any $x \in H$,

$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2$$

In particular, the series on the right-hand-side above is finite for all $x \in H$. 

Tensor products of Hilbert Spaces

Definition. Let $H_1$ and $H_2$ be (non-empty) separable, complex Hilbert spaces. A pair $(H, \otimes)$ is called a tensor product of $H_1$ and $H_2$ if

- $H$ is a Hilbert space
- $\otimes$ is a bilinear map $\otimes : H_1 \times H_2 \to H$, $(\phi_1, \phi_2) \mapsto \phi_1 \otimes \phi_2$
  
  for which the following properties hold

  - For all $\phi_1, \phi_2 \in H_1$ and $\psi_1, \psi_2 \in H_2$,
    
    \begin{equation}
    \langle \phi_1 \otimes \psi_1, \phi_2 \otimes \psi_2 \rangle_H = \langle \phi_1, \phi_2 \rangle_{H_1} \cdot \langle \psi_1, \psi_2 \rangle_{H_2},
    \end{equation}

  - Whenever $\{e_j\}_{j=1}^{\infty}$ is an ONB of $H_1$ and $\{f_k\}_{k=1}^{\infty}$ is an ONB of $H_2$, then $\{e_j \otimes f_k\}_{j,k=1}^{\infty}$ is an ONB of $H$.

Remarks.

A map $B : H_1 \times H_2 \to H$, $(\phi, \psi) \mapsto B(\phi, \psi)$, is said to be bilinear if

- $B(\lambda \phi_1 + \mu \phi_2, \psi) = \lambda B(\phi_1, \psi) + \mu B(\phi_2, \psi)$ for all $\phi_1, \phi_2 \in H_1$, $\psi \in H_2$ and $\lambda, \mu \in \mathbb{C}$.
- $B(\phi, \psi_1 + \mu \psi_2) = \lambda B(\phi, \psi_1) + \mu B(\phi, \psi_2)$ for all $\phi, \psi_1, \psi_2 \in H_2$ and $\lambda, \mu \in \mathbb{C}$.
2) Elements of the form $A \otimes 4 \in H$, where $A \in H_1$ and $4 \in H_2$, are called simple tensors.

3) If $H_1$ and $H_2$ are separable, complex Hilbert spaces and $(H, \otimes)$ is a pair for which
   - $H$ is a Hilbert space
   - $\otimes$ is a bilinear map (as above) for which $\otimes$ holds

   Then, for any OVB $\{e_i\}_{i \in I}$ in $H_1$ and $\{f_j\}_{j \in J}$ in $H_2$, the collection $\{ e_i \otimes f_j \}_{i \in I, j \in J}$ is clearly an ON set.

   In this case, it is not hard to check that

   $\{ e_i \otimes f_j \}_{i \in I, j \in J}$ is an OVB for $H$

   if and only if

   $H = \text{span} \{ e_i \otimes f_j \}_{i \in I, j \in J}$

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**Proposition 1**: Let $H_1$ and $H_2$ be separable, complex Hilbert spaces and $(H, \otimes)$ a corresponding tensor product.

a) For all $A \in H_1$ and $4 \in H_2$,

   $\| A \otimes 4 \|_H = \| A \|_{H_1} \| 4 \|_{H_2}$.
b) If \( \phi_n \to \phi \) in \( H_1 \) and \( 4n \to 4 \) in \( H_2 \), then
\[ \phi \otimes 4n \to 0 \otimes 4 \text{ in } H. \]

c) If the property about ONBs holds for one pair of ONBs, then it holds for any pair of ONBs. In particular, we need only check the 2nd condition on \((H, \otimes)\) for one pair of ONBs.

**Proof.**

a) Let \( \phi \in H_1 \) and \( 4 \in H_2 \), then
\[ \| \phi \otimes 4 \|^2 = \langle \phi \otimes 4, \phi \otimes 4 \rangle_H = \langle \phi, \phi \rangle_{H_1} \cdot \langle 4, 4 \rangle_{H_2} \]
\[ = \| \phi \|^2_{H_1} \cdot \| 4 \|^2_{H_2}. \]

b) Suppose \( \phi_n \to \phi \) in \( H_1 \) and \( 4n \to 4 \) in \( H_2 \), then
\[ \| \phi \otimes 4 - \phi_n \otimes 4n \|^2_H = \| (\phi - \phi_n) \otimes 4 + \phi_n \otimes (4 - 4n) \|^2_H \]
\[ \leq \| \phi - \phi_n \|^2_{H_1} \cdot \| 4 \|^2_{H_2} + \| \phi_n \|^2_{H_1} \cdot \| 4 - 4n \|^2_{H_2} \]
\[ \to 0 \text{ as } n \to \infty. \]

Here we have used bilinearity and the fact that: if \( \phi_n \to \phi \) in \( H_1 \), then \( \| \phi_n \|^2_{H_2} \) is bounded. (Check!)
(c) Let \( \mathbf{e}_{j} \) be an ONB for \( H_{1} \).
Let \( \mathbf{f}_{k} \) be an ONB for \( H_{2} \).

Suppose \( \mathbf{e}_{j} \otimes \mathbf{f}_{k} \) is an ONB for \( H \).

Let \( \mathbf{e}_{j} \) be an ONB for \( H_{1} \).
Let \( \mathbf{f}_{k} \) be an ONB for \( H_{2} \).

By our remark, we need only show that

\[
H = \text{span} \{ \mathbf{e}_{j} \otimes \mathbf{f}_{k} \}_{j,k \in \mathbb{Z}}
\]

In fact, the same remark shows that, by assumption, we know that

\[
H = \text{span} \{ \mathbf{e}_{j} \otimes \mathbf{f}_{k} \}_{j,k \in \mathbb{Z}}
\]

The proof is complete if we then show that:

For each \( j,m \in \mathbb{Z} \), \( \mathbf{e}_{j} \otimes \mathbf{f}_{m} \in \text{span} \{ \mathbf{e}_{j} \otimes \mathbf{f}_{k} \}_{j,k \in \mathbb{Z}} \)

(check this.)

Since \( \mathbf{e}_{j} \) and \( \mathbf{f}_{k} \) are ONBs, it is clear that

\[
\mathbf{e}_{j} = \sum_{j=1}^{N} \mathbf{e}_{j} \quad \text{and} \quad \mathbf{f}_{m} = \sum_{k=1}^{N} \mathbf{f}_{k}
\]

Let \( N \geq 1 \). Clearly

\[
\left( \sum_{j=1}^{N} \mathbf{e}_{j} \otimes \mathbf{e}_{j} \right) \otimes \left( \sum_{k=1}^{N} \mathbf{f}_{k} \otimes \mathbf{f}_{k} \right) = \sum_{j,k \in \mathbb{Z}} \mathbf{e}_{j} \otimes \mathbf{f}_{k} \in \text{span}(H)
\]

By linearity.
Using again that $\bar{\xi}_j^3 \bar{j}_{21}$ and $\bar{\xi}_k^3 \bar{k}_{21}$ are ONB's, it is clear that

$$\sum_{j=1}^{N} \xi_j \bar{\xi}_j \to \xi_0 \quad \text{in } H_1 \quad \text{and} \quad \sum_{k=1}^{N} \delta_k \bar{\xi}_k \to \delta_0 \quad \text{in } H_2$$

and thus

$$\xi_0 \otimes \delta_0 = \lim_{N \to \infty} \left( \sum_{j=1}^{N} \xi_j \bar{\xi}_j \right) \otimes \left( \sum_{k=1}^{N} \delta_k \bar{\xi}_k \right) \in \text{span} \{ \xi_j \otimes \bar{\xi}_j \}_{j=1}^{N} \cap \text{span} \{ \delta_k \otimes \bar{\xi}_k \}_{k=1}^{N},$$

where we have used part b). This completes the proof.

**Proposition 2:** Let $H_1$ and $H_2$ be separable, complex Hilbert spaces.

i) There exists a tensor product $(H \otimes)$ of $H_1$ and $H_2$.

ii) If $(H, \otimes)$ and $(\tilde{H}, \tilde{\otimes})$ are both tensor products of $H_1$ and $H_2$, then there exists an isometric isomorphism $J : H \to \tilde{H}$ such that

$$J(\varphi \otimes \psi) = \varphi \tilde{\otimes} \psi \quad \text{for all } \varphi \in H_1 \text{ and } \psi \in H_2.$$

In this case, we will denote the (essentially) unique tensor product of $H_1$ and $H_2$ by

$$H = H_1 \otimes H_2.$$
Remark:
A linear map \( J : H_1 \to H_2 \) is called an isometric isomorphism if

i) \( J \) is bijective (i.e. 1 to 1 and onto)

ii) \( \langle J \phi_1, J \phi_2 \rangle_{H_2} = \langle \phi_1, \phi_2 \rangle_{H_1}, \) for all \( \phi_1, \phi_2 \in H_1. \)

Proof:
We only consider the case where \( H_1 \) and \( H_2 \) are infinite dimensional. All other cases follow similarly.

i) Take
\[
H = \{ a : \mathbb{N}^2 \to \mathbb{C} \mid \sum_{n,m \geq 1} |a(n,m)|^2 < \infty \}.
\]
(\( \text{Often } H = L^2(\mathbb{N}^2). \)) For homework, we will check that

- One checks that \( H \) is a complex vector space under the usual addition and scalar multiplication of functions.
- One checks that
\[
\langle a, b \rangle = \sum_{n,m \geq 1} \overline{a(n,m)} b(n,m)
\]
defines an inner-product on \( H. \)
- One checks that \( \| a \| = \sqrt{\langle a, a \rangle} = \sqrt{\sum_{n,m \geq 1} |a(n,m)|^2} \) is a complete norm, and hence \( H \) is a Hilbert space.
For any \( j,k \in \mathbb{N} \), \( \rho_{jk} : \mathbb{N}^2 \to \mathbb{R} \) is given by

\[
\rho_{jk}(n,m) = \delta_{jn} \cdot \delta_{km} \quad \text{for all } n,m \geq 1
\]

defines an orthonormal collection of vectors in \( \mathcal{H} \).

Moreover, \( \sum \rho_{jk}\delta_{jk} \) is an OUB of \( \mathcal{H} \).

Let us now define the bilinear map \( \phi \).

Fix \( \sum \rho_{jk}\delta_{jk} \) an OUB in \( \mathcal{H}_1 \) and \( \sum \rho_{jk}\delta_{jk} \) an OUB in \( \mathcal{H}_2 \).

For any \( \phi \in \mathcal{H}_1 \) and \( \psi \in \mathcal{H}_2 \), write

\[
\phi = \sum_{j \in \mathbb{N}} \rho_{j}\phi_j \quad \text{and} \quad \psi = \sum_{k \in \mathbb{N}} \rho_{k}\psi_k.
\]

Define

\[
\phi \otimes \psi = \sum_{j,k \in \mathbb{N}} \rho_{jk} \rho_{j} \otimes \rho_{k} \psi_k.
\]

Claim: \( \phi \otimes \psi \in \mathcal{H} \).

Check

\[
\| \phi \otimes \psi \|_{\mathcal{H}}^2 = \sum_{j,k \in \mathbb{N}} \rho_{jk}^2 \rho_{j}^2 \otimes \rho_{k}^2 = \left( \sum_{j \in \mathbb{N}} \rho_{j}^2 \right) \left( \sum_{k \in \mathbb{N}} \rho_{k}^2 \right)
\]

\[
= \| \phi \|_{\mathcal{H}_1}^2 \cdot \| \psi \|_{\mathcal{H}_2}^2 < \infty.
\]

Thus \( \phi \otimes \psi \in \mathcal{H} \).
The fact that $\otimes$ is bilinear follows from uniqueness of the coefficients in the expansion with respect to an ONB. (check ...)

To see (∗), let $\tilde{\psi} \in \mathcal{H}_1$ and $\tilde{\phi} \in \mathcal{H}_2$.

Expanding as before we see that:

$$
\langle d \otimes \tilde{d}, \tilde{d} \otimes d \rangle = \sum_{j,k \geq 1} \left( \overline{c_j d_k} \cdot \tilde{c}_j \tilde{d}_k \right)
$$

$$
= \sum_{j,k \geq 1} c_j d_k \cdot \overline{\tilde{c}_j \tilde{d}_k}
$$

$$
= \left( \sum_{j \geq 1} \overline{c_j \tilde{c}_j} \right) \left( \sum_{k \geq 1} \overline{d_k \tilde{d}_k} \right)
$$

$$
= \langle d, \tilde{d} \rangle_{\mathcal{H}_1} \cdot \langle \tilde{d}, d \rangle_{\mathcal{H}_2}.
$$

To check the fact about ONBs, note that

$$
e_{j,k} \otimes e_{j,k} = e_{j,k}
$$

Thus for this choice of bases $\{ e_{j,k} \}_{j,k \geq 1}$ in $\mathcal{H}_1$ and $\{ f_{j,k} \}_{j,k \geq 1}$ in $\mathcal{H}_2$

$$
\{ e_{j,k} \otimes f_{j,k} \}_{j,k \geq 1} = \{ e_{j,k} f_{j,k} \}_{j,k \geq 1},
$$

which we know

is an ONB of $\mathcal{H} = l^2(1 \mathbb{N}^2)$. By Proposition 1c), we are done.
To see Proposition 2 ii), let $f \in H_{21}$ and $g \in H_{321}$ be ones in $H_1$ and $H_2$ respectively.

Define a linear map $\mathcal{J} : H \to \tilde{H}$ by setting

$$\mathcal{J}(e_j \otimes f_k) = e_j \otimes \tilde{f}_k \quad \text{for all } j,k \geq 1.$$

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Claim: $\mathcal{J}$ is an isometric isomorphism of $H$ into $\tilde{H}$.

(Proof.)

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**Example 1**

Let $m \geq 1$ and $n \geq 1$ be integers.

Let $H_1 = \mathbb{C}^m$ and $H_2 = \mathbb{C}^n$ equipped with the standard inner products.

Claim: $H_1 \otimes H_2 = \mathbb{C}^{mn}$ equipped with the Hilbert-Schmidt inner product.

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Recall: If $A, B \in \mathbb{C}^{mn}$, then

$$\langle A, B \rangle_{HS} = \text{Tr}[A^* B] = \sum_{j=1}^{m} \sum_{k=1}^{n} a_{j,k}^* b_{j,k} = \sum_{j=1}^{m} \sum_{k=1}^{n} a_{j,k}^* b_{j,k}$$

where $A = [a_{j,k}]$ and $B = [b_{j,k}]$. 

Note that $\text{Tr}[A^* B] = \sum_{j=1}^{m} \sum_{k=1}^{n} a_{j,k}^* b_{j,k}$.

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Regarding vectors $u \in \mathbb{C}^n$ and $v \in \mathbb{C}^n$ as columns, we define $\otimes : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^{mn}$ by setting:

$$u \otimes v = uv^t = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$$

$$= \begin{pmatrix} u_1 v_1 & u_1 v_2 & \cdots & u_1 v_n \\ u_2 v_1 & u_2 v_2 & \cdots & u_2 v_n \\ \vdots & \vdots & \ddots & \vdots \\ u_m v_1 & u_m v_2 & \cdots & u_m v_n \end{pmatrix} \in \mathbb{C}^{mn}.$$  

Bilinearity of $\otimes$ follows from basic matrix arithmetic. Moreover,

$$\langle u \otimes v, \tilde{u} \otimes \tilde{v} \rangle_{\mathbb{C}^{mn}} = \sum_{j=1}^{m} \sum_{k=1}^{n} (u \otimes v)(j,k) \overline{\tilde{u}} \tilde{v}_{j,k}$$

$$= \sum_{j,k} u_k \overline{v}_j \tilde{u}_k \tilde{v}_j$$

$$= \left( \sum_{k=1}^{n} \tilde{u}_k u_k \right) \left( \sum_{j=1}^{m} \tilde{v}_j v_j \right)$$

$$= \langle u, \tilde{u} \rangle_{\mathbb{C}^n} \langle v, \tilde{v} \rangle_{\mathbb{C}^n}.$$
Finally, if we let \( \{ e_j \}_{j \in \mathbb{J}} \) and \( \{ f_k \}_{k \in \mathbb{K}} \) be the standard basis vectors in \( \mathbb{C}^m \) and \( \mathbb{C}^n \) respectively, then one readily checks that

\[
e_j \otimes f_k = e_j f_k^t = E_{jk}
\]

where \( E_{jk} \) is the matrix with 1 in the \( (j,k) \) entry and 0 everywhere else. It is clear that

\[
\bigoplus E_{jk} e_j \otimes f_k\bigoplus \quad \text{is an ONB of } \mathbb{C}^{mn}.
\]

**Example 2**

Let \((X, d_x)\) and \((Y, d_y)\) be \(T\)-finite measure spaces.

It is known that

\( H_1 = L^2(X, d_x) \) and \( H_2 = L^2(Y, d_y) \) are separable, complex Hilbert spaces.

**Claim:** \( H_1 \otimes H_2 = L^2(X \times Y, d(x) d(y)) \).

To see this, let \( \phi \in L^2(X, d_x) \) and \( \psi \in L^2(Y, d_y) \).

Define

\[
(\phi \otimes \psi)(x, y) = \phi(x) \cdot \psi(y).
\]

All relevant properties can be checked.