Theorem. Let $H_1$ and $H_2$ be separable complex Hilbert spaces. For any $A \in B(H_1)$ and $B \in B(H_2)$, there is a unique linear operator $A \otimes B \in B(H_1 \otimes H_2)$ for which

\[ (A \otimes B)(\Omega \otimes \Phi) = (A \Phi) \otimes (B \Omega) \]

holds for all $\Phi \in H_1$ and $\Omega \in H_2$. Moreover,

\[ \|A \otimes B\| = \|A\| \cdot \|B\| \].

Proof:

We set

\[ M = \text{Span}\{ \Omega \otimes \Phi : \Phi \in H_1 \text{ and } \Omega \in H_2 \}. \]

We saw that $M \subset H_1 \otimes H_2$ is a dense subspace.

Using $\square$, we defined a linear map $C_{AB} : M \rightarrow H_1 \otimes H_2$ and we checked that

1) $C_{AB}$ is well-defined

2) $C_{AB} \in B(M)$ with $\|C_{AB}\| \leq \|A\| \cdot \|B\|$. 

We will now show

\[ \|C_{AB}\| \geq \|A\| \cdot \|B\|. \]
To see this, we argue as follows.

Note: If \( A = 0 \) or \( B = 0 \), then \( \|CA, B\| = 0 \) by ii).
In this case, we are done.

Otherwise, \( A \neq 0 \) and \( B \neq 0 \), and \( 0 < \min[\|A\|_1, \|B\|_1] \).

For any \( 0 < \varepsilon < \min[\|A\|_1, \|B\|_1] \), \( \exists \phi_\varepsilon \in \mathcal{H}_1 \) and \( \psi_\varepsilon \in \mathcal{H}_2 \)
for which

\[
\|A \phi_\varepsilon\|_1 \geq \|A\|_1 - \varepsilon, \quad \|B \psi_\varepsilon\|_1 \geq \|B\|_1 - \varepsilon, \quad \text{and} \quad \|\phi_\varepsilon\|_1 = \|\psi_\varepsilon\|_1 = 1.
\]

(This uses that \( \|A\|_1 = \sup_{\|\phi\|_1 = 1} \frac{\|A\phi\|_1}{\|\phi\|_1} \).)

\[
\Rightarrow \|CA, B\| \geq \|C(A \phi_\varepsilon, (B \psi_\varepsilon)\|_1
= \|A \phi_\varepsilon \otimes (B \psi_\varepsilon)\|_1
= \|A \phi_\varepsilon\|_1 \cdot \|B \psi_\varepsilon\|_1
\geq (\|A\|_1 - \varepsilon)(\|B\|_1 - \varepsilon)
\]

Taking limit as \( \varepsilon \to 0 \), we see that \( \|CA, B\| \geq \|A\|_1 \cdot \|B\|_1 \)
and we are done.
More properties of tensor products of operators.

Proposition 1: Let $H_1$ and $H_2$ be separable complex Hilbert spaces.

a) The map $(\cdot, \cdot) : B(H_1) \times B(H_2) \to B(H_1 \otimes H_2)$

given by $(A, B) \mapsto A \otimes B$ is bilinear.

b) For all $A_1, A_2 \in B(H_1)$ and $B_1, B_2 \in B(H_2)$,

$$(A_1 A_2) \otimes (B_1 B_2) = (A_1 \otimes B_1) (A_2 \otimes B_2).$$

c) For all $A \in B(H_1)$ and $B \in B(H_2)$,

$$(A \otimes B)^* = A^* \otimes B^*.$$  

d) If $A_n \to A$ in $B(H_1)$ and $B_n \to B$ in $B(H_2)$,

then $A_n \otimes B_n \to A \otimes B$ in $B(H_1 \otimes H_2)$.

The proofs of parts a) and part d) are homework.

Note: Given b), if $A$ and $B$ are invertible, then $A \otimes B$ is invertible. In fact,

$$(A \otimes B) (A^* \otimes B^*) = I_1.$$  (Here need "boundedly invertible")

More on this in the homework.
Proof:

b) Let \( \psi \in \mathcal{H}_1 \) and \( \eta \in \mathcal{H}_2 \).

Note that

\[
\begin{align*}
[(A_1 \otimes B_1) \otimes (A_2 \otimes B_2)](\psi \otimes \eta) &= (A_1 \otimes A_2) \otimes (B_1 \otimes B_2) \\
&= [(A_1 \otimes B_1)](A_2 \otimes B_2) \\
&= [(A_1 \otimes B_1)(A_2 \otimes B_2)](\psi \otimes \eta)
\end{align*}
\]

The above clearly extends to finite linear combinations and hence to the dense set \( \mathcal{M} = \text{span}\{\psi \otimes \eta : \psi \in \mathcal{H}_1, \eta \in \mathcal{H}_2\} \).

Thus these two bounded linear operators agree on a dense subspace. By the BLT Theorem, there is a unique extension to \( \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) \), and hence these operators agree.

Let \( \phi, \tilde{\eta} \in \mathcal{H}_1 \) and \( \eta, \tilde{\psi} \in \mathcal{H}_2 \).

Note that

\[
\begin{align*}
\langle (A \otimes B)^* (\eta \otimes \psi), \tilde{\eta} \otimes \tilde{\psi} \rangle &= \langle \phi \otimes \psi, (A \otimes B)(\eta \otimes \psi) \rangle \\
&= \langle \phi \otimes \psi, (A \otimes B)(\eta \otimes \psi) \rangle \\
&= \langle \phi, A \tilde{\psi} \rangle \cdot \langle \eta, B \tilde{\psi} \rangle \\
&= \langle A^* \phi, \tilde{\psi} \rangle \cdot \langle \eta, B^* \tilde{\psi} \rangle \\
&= \langle A^* \phi \otimes B^* \eta, \tilde{\psi} \otimes \tilde{\psi} \rangle \\
&= \langle (A^* \otimes B^*)^* (\phi \otimes \psi), \tilde{\psi} \otimes \tilde{\psi} \rangle
\end{align*}
\]
From this, we conclude that

- For each fixed \( \omega \in H_1 \) and \( \nu \in H_2 \), the above equality holds for all \( \omega \in H_1 \) and \( \nu \in H_2 \).

- Thus the equality holds for all \( \omega \otimes \nu \in H_1 \otimes H_2 \) and for all elements of
  \[ \mathcal{M} = \text{span} \{ \omega \otimes \nu \} \]
  a dense subspace.

- This implies that
  \[ (A \otimes B)^* (\omega \otimes \nu) = (A^* \otimes B^*) (\omega \otimes \nu) \]
  for all \( \omega \in H_1 \) and \( \nu \in H_2 \). (As vectors)

- Arguing as before, these two bounded linear operators must then agree by the BCT Theorem.

Let us now consider self-adjoint operators.

By the previous proposition, if \( A \) and \( B \) are self-adjoint, then \( A \otimes B \) is also self-adjoint. In general, the converse is not true. Here is a fact.

**Proposition 2:** Let \( H_1 \) and \( H_2 \) be separable complex Hilbert spaces.

Let \( A \in \mathcal{B}(H_1) \) and \( B \in \mathcal{B}(H_2) \).

a) If \( A \otimes B \) is self-adjoint and \( A \) (or \( B \)) is self-adjoint, then \( B \) (or \( A \)) is also self-adjoint.
b) If $A \otimes B \neq 0$ is self-adjoint, then there is a number $c \in \mathbb{C}/503$ for which

$$\hat{A} = cA \quad \text{and} \quad \hat{B} = \frac{1}{c}B$$

are self-adjoint operators.

**Proof:**

a) Suppose $A \otimes B \neq 0$ is self-adjoint and $B$ is self-adjoint.

Note: $B \neq 0$ (since otherwise $A \otimes B = 0$) and thus there is $\eta \in \mathbb{H}/503$ for which $B \eta \neq 0$.

**Note for that:** For any $\psi \in \mathbb{H}$,

$$(A \psi) \otimes (B \eta) = (A \otimes B)(\psi \otimes \eta) = (A \otimes B)^\ast(\psi \otimes \eta)$$

$$= (A^\ast \otimes B^\ast)(\psi \otimes \eta)$$

$$= (A^\ast \psi) \otimes (B^\ast \eta)$$

$$\Rightarrow (A^\ast \psi) \otimes (B \eta) = 0$$

$$\Rightarrow A \psi = A^\ast \psi \quad \text{for all } \psi \in \mathbb{H},$$

$$\Rightarrow A \text{ is self-adjoint.}$$

The other case is almost identical.
b) Suppose \( A \otimes B \neq 0 \) is self-adjoint.

If \( A \) (or \( B \)) is self-adjoint, then by the previous result this holds with the choice of \( c=1 \).

Let us now assume that neither \( A \) nor \( B \) are self-adjoint.

Choose \( \psi \) for which

\[
\begin{align*}
&\psi \neq 0 \quad \text{(true since } B \neq 0) \\
&\langle \psi, \psi \rangle \neq 0 \quad \text{(check!)}
\end{align*}
\]

For any such choice, let \( d_1, d_2 \in H_1 \).

Note that

\[
\begin{align*}
\langle A d_1, d_2 \otimes \psi \rangle &= \langle A d_1 \otimes \psi, d_2 \otimes \psi \rangle \\
&= \langle A \otimes B \psi, (d_1 \otimes \psi) \otimes (d_2 \otimes \psi) \rangle \\
&= \langle d_1 \otimes \psi, (A \otimes B) (d_2 \otimes \psi) \rangle \\
&= \langle d_1, A \psi \rangle \langle \psi, B \psi \rangle
\end{align*}
\]

Let \( \langle \psi, B \psi \rangle \neq 0 \); we conclude that

\[
\langle cA d_1, d_2 \rangle = \langle d_1, cA \psi \rangle \quad \text{for all } d_1, d_2 \in H_1,
\]

\( \Rightarrow \) \( A = cA \) is self-adjoint.

Moreover, since

\[
A \otimes B = \widetilde{A} \otimes B \quad \text{with } \widetilde{B} = \frac{1}{c} B
\]

we conclude that \( B \) is self-adjoint by part a).
The following is a useful observation.

**Proposition 3**

Let \( A \in \text{BC} (\mathbb{C}^m) \) and \( B \in \text{BC} (\mathbb{C}^n) \) both be self-adjoint.

Let \( \xi \xi_j \), \( j = 1, \ldots, m \), and \( \xi \eta_k \eta_k \), \( k = 1, \ldots, n \), be the eigenvalues of \( A \) and \( B \) respectively, each counted according to multiplicity.

Then:

1. The self-adjoint operator \( A \otimes B \) has eigenvalues \( \xi \eta_j \eta_k \), \( j = 1, \ldots, m \), \( k = 1, \ldots, n \).

2. The self-adjoint operator

\[
A \otimes I + I \otimes B
\]

has eigenvalues \( \xi \eta_j + \eta_k \), \( j = 1, \ldots, m \), \( k = 1, \ldots, n \).

**Proof:**

By the spectral theorem, there are orthonormal bases

\( \{ \xi_j \} \), \( j = 1, \ldots, m \), and \( \{ \eta_k \} \), \( k = 1, \ldots, n \), of \( \mathbb{C}^m \) and \( \mathbb{C}^n \) respectively, for which:

\[
A = \sum_{j=1}^{m} \xi_j \xi_j^* = \sum_{j=1}^{m} \xi_j \xi_j^* \eta_j \eta_j^* \\
B = \sum_{k=1}^{n} \eta_k \eta_k^* = \sum_{k=1}^{n} \eta_k \eta_k^* \xi_k \xi_k^*
\]

and

\[
A \otimes I + I \otimes B = \sum_{j=1}^{m} \xi_j \xi_j^* \otimes I + I \otimes B = \sum_{j=1}^{m} \xi_j \xi_j^* \otimes \sum_{k=1}^{n} \eta_k \eta_k^* + \sum_{j=1}^{m} \xi_j \xi_j^* \otimes \sum_{k=1}^{n} \eta_k \eta_k^* \\
= \sum_{j=1}^{m} \sum_{k=1}^{n} \xi_j \eta_k \eta_k \xi_j \xi_j^* = \sum_{j=1}^{m} \sum_{k=1}^{n} \xi_j \eta_k \eta_k \xi_j \xi_j^* \\
= \sum_{j=1}^{m} \sum_{k=1}^{n} \xi_j \eta_k \eta_k \xi_j \xi_j^* = \sum_{j=1}^{m} \sum_{k=1}^{n} \xi_j \eta_k \eta_k \xi_j \xi_j^*
\]
It is then clear that $\sum_{j,k} e_j \otimes f_k \delta_{j,k}^{j_2 k_2}$ is an orthonormal basis for $C^m \otimes C^n$.

Moreover,

$$(A \otimes B)(e_j \otimes f_k) = \delta_{j,k} A(e_j) \otimes B(f_k)$$

and

$$[A \otimes \mathbb{I} + \mathbb{I} \otimes B](e_j \otimes f_k) = (A + \mathbb{I} B)(e_j \otimes f_k)$$

with each relation holding for all $1 \leq j \leq m$ and $1 \leq k \leq n$.

Since $\sum_{j,k} e_j \otimes f_k \delta_{j,k}^{j_2 k_2}$ is an OBP, this is a complete description of the spectrum of these operators.

Note: We also know the corresponding eigenvectors $\mathbb{I}$.

Another useful observation about the tensor product of operators:

**Theorem:** Let $H_1$ and $H_2$ be separable complex Hilbert spaces. Let $A \in B(H_1)$ and $B \in B(H_2)$. Define $H$ by

$$H = A \otimes \mathbb{I} + \mathbb{I} \otimes B \in B(H_1 \otimes H_2).$$

One has that

$$e^H = e \otimes e.$$
The above theorem holds for arbitrary bounded calculus $A$ and $B$.

Recall: If $\mathcal{H}$ is a Hilbert space and $A \in \mathcal{B}(\mathcal{H})$, then

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$ 

This series converges in $\mathcal{B}(\mathcal{H})$.

When $A$ is self-adjoint, this definition coincides with the operator defined via functional calculus.

This is how the equation

$$e^A = e^A \otimes e^B$$

is to be understood.

An immediate corollary of this theorem is:

**Corollary** Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be separable complex Hilbert spaces.

Let $A \in \mathcal{B}(\mathcal{H}_1)$, with $A$ self-adjoint.

Let $B \in \mathcal{B}(\mathcal{H}_2)$, with $B$ self-adjoint.

For any $t \in \mathbb{R}$,

$$e^{-itH} = e^{-itA} \otimes e^{-itB}$$

where $H = A \otimes 1 + 1 \otimes B$. 

In words, this corollary says that the Schrödinger evolution corresponding to $H$ is just the tensor product of the Schrödinger evolutions of $A$ and $B$. This is because $H = A \otimes 4l + 4 \otimes (-HB)$ is a "non-interacting" Hamiltonian.

The proof of the corollary is simple. Note that for any $t \in \mathbb{R}$

$$ -itH = (-itA) \otimes 4l + 4 \otimes (-itB) $$

and both $(-itA)$ and $(-itB)$ are bounded if $A$ and $B$ are bounded and $t$ is fixed.

To prove the theorem, first recall a simple fact.

For any $x, y \in \mathbb{R}$,

$$ e^{x+y} = e^x \cdot e^y. $$

One can prove this with Taylor series:

$$ e^{x+y} = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{(n)}{k!} x^k y^{n-k} $$

Using the binomial expansion,

$$ = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{1}{k!} \frac{A^k}{k!(nt)!} x^k y^{n-k} $$

$$ = \sum_{k=0}^{\infty} x^k \cdot \sum_{m=0}^{\infty} \frac{y^m}{m!} = e^x \cdot e^y. $$
We use this fact as follows.

Let $\xi, \eta \in \mathcal{B}(H)$ and suppose $[\xi, \eta] = 0$ (i.e. $\xi \eta = \eta \xi$.)

\[
\xi \eta = \xi + \eta = \xi - \eta
\]

(Since the bounded operators commute, the analogue of the binomial expansion holds. In this case, the previous argument holds for operators too!)

We apply this as follows.

Let $\xi = A \otimes 1$ and $\eta = 1 \otimes B$.

\[
\xi \eta = (A \otimes 1)(1 \otimes B) = A \otimes B = (1 \otimes B)(1 \otimes A) = \eta \xi.
\]

Thus

\[
H = A \otimes 1 \otimes B
\]
\[
e = e \cdot 1 \otimes e
\]

We will now show that

\[
e \otimes 1 = e \otimes 1 \quad \text{and similarly} \quad 1 \otimes e
\]

Given this

\[
e = e \cdot e = (e \otimes 1)(1 \otimes e) = A \otimes e
\]

and we are done!
We prove this result for $A \otimes 1$.

The result for $1 \otimes B$ is quite similar.

$$
\sum_{n=0}^{\infty} \frac{(A \otimes 1)^n}{n!} = \lim_{N \to \infty} \sum_{n=0}^{N} \frac{(A \otimes 1)^n}{n!} = \lim_{N \to \infty} \left( \sum_{n=0}^{N} \frac{A^n}{n!} \right) \otimes 1 = e^A \otimes 1
$$

This completes the proof.