(1) Consider the collection of functions
\[ H = \{ a : \mathbb{N}^2 \to \mathbb{C} \text{ with } \sum_{n,m \geq 1} |a(n,m)|^2 < \infty \} \]
This set is often written as \( H = \ell^2(\mathbb{N}^2) \). (The letter \( a \) is used for the functions since they are often thought of as sequences . . .) The goal of this exercise is to show that \( H \) is a Hilbert space with an explicit orthonormal basis.

i) Show that \( H \) is a complex vector space under the usual rules of arithmetic for functions. Moreover, show that
\[ \langle a, b \rangle = \sum_{n,m \geq 1} a(n,m) \overline{b(n,m)} \]
for all \( a, b \in H \) is an inner product on \( H \).

ii) Show that \( H \), equipped as above, is a Hilbert space.

iii) Show that the collection of functions \( \{ e_{jk} \}_{j,k \geq 1} \) with \( e_{jk}(n,m) = \delta_{jn} \cdot \delta_{km} \) is an orthonormal basis of \( H \).

(2) Let \( H \) be a separable Hilbert space and \( \{ e_j \}_{j \geq 1} \subset H \) be an orthonormal set. Prove that \( \{ e_j \}_{j \geq 1} \) is an orthonormal bases of \( H \) if and only if
\[ H = \overline{\text{span}(e_j : j \geq 1)} \].

(3) Let \( H_1 \) and \( H_2 \) be separable Hilbert spaces. Let \( M \subset H_1 \otimes H_2 \) be the following dense subspace:
\[ M = \overline{\text{span}(\phi \otimes \psi : \phi \in H_1 \text{ and } \psi \in H_2)} \].
We showed in class that the linear map \( C_{1,B} : M \to H_1 \otimes H_2 \) defined by requiring
\[ C_{1,B}(\phi \otimes \psi) = \phi \otimes B\psi \]
for all \( \phi \in H_1 \) and \( \psi \in H_2 \)
is well-defined. Show that $\|C_{\mathbb{L}, \mathbb{B}}\| \leq \|B\|$ and carefully describe each line in the estimate.

(4) Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be separable Hilbert spaces.

i) Show that the map $(\cdot, \cdot) : \mathcal{B}(\mathcal{H}_1) \times \mathcal{B}(\mathcal{H}_2) \to \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ with $(A, B) \mapsto A \otimes B$ is bilinear.

ii) Show that if $A_n \to A$ in $\mathcal{B}(\mathcal{H}_1)$ and $B_n \to B$ in $\mathcal{B}(\mathcal{H}_2)$, then $A_n \otimes B_n \to A \otimes B$ in $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$.

iii) Let $A \in \mathcal{B}(\mathcal{H}_1)$ and $B \in \mathcal{B}(\mathcal{H}_2)$. Show that $A \otimes B$ is boundedly invertible if and only if $A$ and $B$ are both boundedly invertible.

**Recall:** $A \in \mathcal{B}(\mathcal{H})$ is boundedly invertible if $A^{-1}$ exists and $A^{-1} \in \mathcal{B}(\mathcal{H})$.

**Hint:** You may want to use that: If $A \in \mathcal{B}(\mathcal{H})$ is boundedly invertible then there is some positive number $C$ for which $\|x\| \leq C\|Ax\|$ for all $x \in \mathcal{H}$.

(5) Let $A \in \mathcal{B}(\mathbb{C}^m)$ and $B \in \mathcal{B}(\mathbb{C}^n)$. We proved in class that $\|A \otimes B\| = \|A\|\|B\|$ where the norm here is the operator norm. Check that the following is also true:

i) For $A$ and $B$ as above, $\text{Tr}[A \otimes B] = \text{Tr}[A]\text{Tr}[B]$.

ii) For $A$ and $B$ as above, $\|A \otimes B\|_1 = \|A\|_1\|B\|_1$.

iii) For $A$ and $B$ as above, $\|A \otimes B\|_2 = \|A\|_2\|B\|_2$.

**Recall:** For any matrix $A \in \mathcal{B}(\mathbb{C}^n)$, the trace norm of $A$, which we denote by $\|A\|_1$, is given by

$$\|A\|_1 = \sum_{j=1}^{\infty} \sigma_j(A)$$

where the numbers being added above are the singular values of $A$. Moreover, for any matrix $A \in \mathcal{B}(\mathbb{C}^n)$, the Hilbert-Schmidt norm of $A$, which we denote by $\|A\|_2$, is given by

$$\|A\|_2 = \sqrt{\text{Tr}[A^*A]}.$$
Using the above as a definition, it is not hard to check that

$$\| A \|_2 = \sqrt{\sum_{j=1}^{n} \sigma_j(A)^2}.$$