(1) Consider the metric space \((\mathbb{Z}, d)\) with 
\[d(x, y) = |x - y|\]
for all \(x, y \in \mathbb{Z}\). 
Show that for any \(a \geq 0\), the function \(F_a : [0, \infty) \to (0, \infty)\) given by 
\[F_a(r) = e^{-ar}\]
for all \(r \geq 0\), 
is not an \(F\)-function on \((\mathbb{Z}, d)\). \textbf{Note:} This result readily generalizes to any \((\mathbb{Z}^\nu, d)\) with \(\nu \geq 1\), but focus in the one-dimensional case.

(2) i) Let \(f : [0, \infty) \to [0, \infty)\) be a continuous function. 
Show that for each \(n \geq 1\), 
\[
\int_0^x \int_0^{t_1} \cdots \int_0^{t_{n-1}} f(t_1) f(t_2) \cdots f(t_n) dt_n dt_{n-1} \cdots dt_1 = \left( \int_0^x f(t) \, dt \right)^n \frac{n!}{n^n}
\]
\textbf{Hint:} Prove this by induction with \(n = 2\) being the base case.
ii) Let \(\mathcal{H}\) be a Hilbert space and \(A : [0, \infty) \to \mathcal{B}(\mathcal{H})\) be norm-continuous. Show that the function \(W : [0, \infty) \to \mathcal{B}(\mathcal{H})\) given by 
\[
W(x) = \mathbf{1} + \sum_{n=1}^{\infty} \int_0^x \int_0^{t_1} \cdots \int_0^{t_{n-1}} A(t_1) A(t_2) \cdots A(t_n) \, dt_n \, dt_{n-1} \cdots \, dt_1
\]
is well-defined for all \(x \in [0, \infty)\) and provide an estimate on \(\|W(x)\|\).

(3) In this problem, we investigate the Gronwall inequality.

i) Let \([a, b] \subset \mathbb{R}\). Suppose \(f : [a, b] \to \mathbb{R}\) is continuous and 
\[
f(x) \leq \alpha + \int_a^x g(t) f(t) \, dt 
\]
for all \(x \in [a, b]\), 
where \(\alpha \in \mathbb{R}\) and \(g : [a, b] \to [0, \infty)\) is continuous. Show that 
\[
f(x) \leq \alpha e^{G(x)}
\]
for all \(x \in [a, b]\) where \(G(x) = \int_a^x g(t) \, dt\).
This estimate is called a Gronwall inequality.

ii) Use this Gronwall inequality to show that the time-dependent Schrödinger equation has a unique solution. More precisely, let \(I \subset \mathbb{R}\) be an open interval and \(\mathcal{H}\) be a complex Hilbert space. Let \(H : I \to \mathcal{B}(\mathcal{H})\) satisfy
\[ H(t)^* = H(t) \] for all \( t \in I \), and
\[ t \mapsto H(t) \] is norm continuous on \( I \).

Then show that the \( \mathcal{H} \)-valued initial value problem:

Let \( t_0 \in I \) and consider
\[
\frac{d}{dt}\psi(t) = -iH(t)\psi(t) \quad \text{with } \psi(t_0) = \psi_0 \in \mathcal{H}
\]
has a unique solution.

ii') A very similar argument show that the \( \mathcal{B}(\mathcal{H}) \)-valued initial value problem:

Let \( t_0 \in I \) and consider
\[
\frac{d}{dt}U(t) = -iH(t)U(t) \quad \text{with } U(t_0) = \mathbb{I} \in \mathcal{B}(\mathcal{H})
\]
also has a unique solution. Check this.

iii) Show that the \( \mathcal{B}(\mathcal{H}) \)-valued initial value problem corresponding to the norm-preservation lemma has a unique solution. More precisely, let \( I \subset \mathbb{R} \) be an open interval and \( \mathcal{H} \) be a complex Hilbert space.

Let \( A, B : I \rightarrow \mathcal{B}(\mathcal{H}) \) satisfy
\[ A(t)^* = A(t) \] for all \( t \in I \), and
\[ \text{both } t \mapsto A(t) \text{ and } t \mapsto B(t) \] are norm continuous on \( I \).

Then show that the \( \mathcal{B}(\mathcal{H}) \)-valued initial value problem:

Let \( t_0 \in I \) and consider
\[
\frac{d}{dt}f(t) = -i[A(t), f(t)] + B(t) \quad \text{with } f(t_0) = f_0 \in \mathcal{B}(\mathcal{H})
\]
has a unique solution.