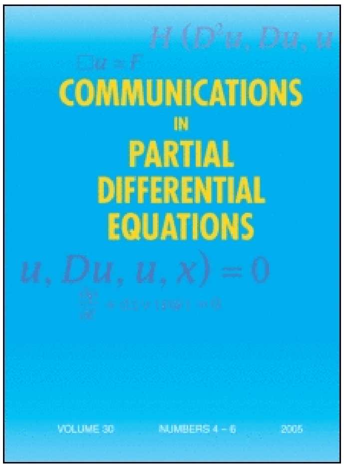

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Regularity Near a Contact Point for Flame Propagation

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Regularity Near a Contact Point for Flame Propagation

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In this paper we study a free boundary problem, arising from a model for the propagation of laminar flames. Consider a cylindrical region S in $\mathbb{R}^n$, and the following free boundary problem with Dirichlet data on ∂S: $u_t = \Delta u$ in $\{u > 0\} \cap S$, $|\nabla u| = 1$ on $\partial\{u > 0\} \cap S$ and $u = 0$ on ∂S. We show that if there is a contact point of the free boundary $\{u = 0, |\nabla u| = 1\}$ with ∂S, then the free boundary approaches ∂S tangentially and it turns out to be a graph of $C^{1+\alpha,\alpha}$ function near the contact point. In particular, the space normal is Hölder continuous.

Keywords Combustion; Contact point; Free-boundary problem; Heat equation; Regularity.

Mathematics Subject Classification 35K05; 35K55; 80A25.

1. Introduction

In this paper we study a parabolic free boundary problem, which appears in combustion theory in the analysis of the propagation of equidiffusional premixed flames with high activation energy. The classical formulation is as follows. Let u_0 be a continuous and nonnegative initial function defined in $\mathbb{R}^n$, whose positive set is open and nonempty. We find a nonnegative continuous function u in $\mathbb{R}^n \times (0, \infty)$ such that

$$\begin{cases} u_t = \Delta u & \text{in } \{u > 0\} \\ |\nabla u| = 1 & \text{on } \partial\{u > 0\} \\ u(x, 0) = u_0(x) & \end{cases} \tag{P}$$

where ∇u denotes the spatial gradient of u and $\{u > 0\}$ denotes the inverse image $\{(x, t) : u(x, t) > 0\}$. In combustion theory for laminar flames, $u = \lambda(T_c - T)$ where T_c is the flame temperature and λ is a normalization factor (See [4, 5, 10]).

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There have been several approaches to construct a solution to (P). In [5], Caffarelli and Vazquez proved uniform estimates for (P_ϵ) and obtained a convergence of the approximating solutions u_ϵ of

$$\begin{cases} \partial_t u_\epsilon = \Delta u_\epsilon - \beta_\epsilon(u_\epsilon) \\ u_\epsilon(x,0) = u_{0\epsilon}(x). \end{cases} \tag{P_ϵ}$$

Here $\beta_\epsilon(z) = \frac{1}{\epsilon}\beta(\frac{z}{\epsilon})$ and $\beta : \mathbb{R} \to \mathbb{R}$ is a C^∞ function which is positive on $(0, 1)$ and 0 otherwise, increasing on $(0, 1/2)$, decreasing on $(1/2, 1)$, and satisfies $\int_0^1 \beta = 1/2$. The initial functions $u_{0\epsilon}$ are C^∞-smooth and nonnegative, $u_{0\epsilon}$ uniformly approximate u_0, and their supports converge to $\{u_0 > 0\}$. Note that (P_ϵ) admits a unique classical solution in $\mathbb{R}^n \times (0, \infty)$ and the maximum principle holds.

Assuming u_0 is bounded and Lipschitz continuous, it was proven (Theorem 7.1, [5]) that u_ϵ converges along subsequences to a function u in $C^{1,1/2}_{loc}$. We call this function a *limit solution* of (P). In its positivity set the limit solution u is a solution of the heat equation. By [8], a limit solution is also a *viscosity solution* and hence the comparison principle holds. Further assuming that u_0 is strictly mean concave in the interior of its support, i.e., for a solution with shrinking support, the limit solution u solves (P) in the sense of distributions [5], and the uniqueness follows from that for viscosity solutions [8]. (Since the heat equation with initial data u_0 is also a viscosity solution, Kim [8] proves the uniqueness for solutions satisfying $\overline{\{u > 0\}} \cap \{t = 0\} = \overline{\{u_0 > 0\}}$.) For more general initial data $u_0 \in C^{0,1}(\mathbb{R}^n)$, Weiss [11] showed that each limit of u_ϵ is a solution of (P) in the sense of domain variation, and for domain variation solutions (u, χ), Andersson and Weiss [1] proved that one-sided flatness of the free boundary implies regularity. In particular, the free boundary can be decomposed into an open regular set with Hölder continuous space normal, and a closed singular set.

In this paper, we study a limit solution u in $S \times (0, \infty)$, where S is the unit cylinder in $\mathbb{R}^n$ with the Dirichlet boundary condition imposed on $\partial S \times (0, \infty)$. The behavior and regularity of the free boundary of u will be investigated near its contact point with ∂S. (The regularity of a solution near a contact point has been studied before for elliptic problems [6] but the free boundary regularity remained open. For obstacle-type elliptic/parabolic problems which has strong regularity and scaling properties, the regularity of free boundary has been obtained in [2, 9] near a contact point.)

The main results in this paper – Theorem 1.1 can be obtained in a local region (for a solution u defined in a local region). Let S be the unit cylinder in $\mathbb{R}^n$, i.e., $S = B_1(0) \times \mathbb{R} \subset \mathbb{R}^n$ and let Q_{r_0} be a parabolic cube in $\mathbb{R}^{n+1}$ such that

$$Q_{r_0} = B_{r_0}(z) \times (0, r_0^2) \subset \mathbb{R}^{n+1} \quad \text{where } z \in \partial S, \;\; r_0 < 1.$$

Let u solve the following free boundary problem in Q_{r_0}

$$\begin{cases} u_t = \Delta u & \text{in } \{u > 0\} \\ |\nabla u| = 1 & \text{on } \partial\{u > 0\} \cap (S \times (0, \infty)) \\ u = 0 & \text{on } \partial S \times (0, \infty). \end{cases} \tag{PS}$$

- Denote by $\Omega_t(u)$, the positive set of u at time t, i.e.,

$$\Omega_t(u) = \{x : u(x, t) > 0\}.$$

- Denote by $\Gamma_t(u)$, the free boundary of u at time t such that

$$\Gamma_t(u) = \overline{\partial\Omega_t(u) \cap S}.$$

- For $(x, t) \in \overline{S} \times (0, \infty)$ and $r > 0$, $Q_r(x, t)$ is the parabolic cube with radius r, and $Q_r^-(x, t)$ is its negative part, i.e.,

$$Q_r(x, t) = B_r(x) \times (t - r^2, t + r^2), \quad Q_r^-(x, t) = B_r(x) \times (t - r^2, t).$$

- For $x \in \partial S$ let ν_x be the inward unit normal to ∂S at x. For $0 < \theta < \pi/2$ let $\Lambda(x, \theta)$ be a cone with vertex at x, opening angle 2θ, and with axis ν_x,

$$\Lambda(x, \theta) = \{x + y : y \cdot \nu_x \geq |y| \cos\theta\} \cap S.$$

If u solves (PS) in $S \times (0, \infty)$ with a bounded Lipschitz continuous initial data u_0, it can be shown as in [5] (or as in [11] for compactly supported u_0) that $u \in C^{1,1/2}_{\text{local}}$ (see Remark 2 for detailed explanation). In the main theorem below, we supposed that u is $C^{1,1/2}(Q_{r_0})$ instead of imposing any assumption on the initial data u_0.

Theorem 1.1. *Let x be a contact point of $\Gamma_t(u)$ with ∂S, i.e., $x \in \Gamma_t(u) \cap \partial S$ and suppose $u \in C^{1,1/2}(Q_{r_0})$ with a norm $A > 0$.*

(A) *For any $0 < \theta < \pi/2$, there exists a small $r > 0$ such that*

$$\Lambda(x, \theta) \cap B_r(x) \subset \Omega_t(u).$$

(B) *There exists $\sigma_n = \sigma_n(A, n) > 0$ such that if*

$$|\nabla u| \leq 1 + \sigma_n \quad \text{on } \partial S \cap Q_\rho^-(x, t) \tag{1.1}$$

then for some $r = r(\rho, A, n) > 0$, any connected component of $\Gamma_t(u) \cap B_r(x)$ is a graph of $C^{1+\alpha,\alpha}$-function, and the normal ν is Hölder continuous.

(C) *If u is decreasing in time, then (1.1) holds for some $\rho > 0$.*

Remark 1. In Theorem 1.1, the unit cylinder S can be replaced by a cylindrical region S' with C^2-boundary.

Remark 2. The proof of $C^{1,1/2}$-regularity in [11] can be modified as below. Let u_0 be a Lipschitz function in S with a compact support. Let L be the Lipschitz constant of u_0 and let $\text{supp}(u_0) \subset B_M(0) \cap S$. For $\delta \in (0, 1)$ and $R > M + (2 \sup u_0/\delta)$, let u_R be the solution of

$$\begin{cases} \partial_t u_R - \Delta u_R = -\beta_\epsilon(u_R) & \text{in } B_R(0) \cap S \times (0, \infty) \\ u_R = u_{0\epsilon} & \text{at } t = 0 \\ u_R = 0 & \text{on } \partial(B_R(0) \cap S) \times (0, \infty). \end{cases}$$

If one can show

$$\sup |\nabla u_R|^2 \leq \delta + 1 + 4L^2 \tag{1.2}$$

then by sending $\delta \to 0$ and $R \to \infty$ it will follow that

$$\sup |\nabla u_\epsilon|^2 \leq 1 + 4L^2. \tag{1.3}$$

For (1.2) it suffices to prove (i) $|\nabla u_R| \leq \delta$ on $\partial B_R(0) \cap S$; (ii) $|\nabla u_R| \leq 2L$ on $\partial S \cap B_R(0)$ (see the proof of Proposition 3.1 of [11]). The proof of (i) is similar to that in [11]. For (ii), define

$$w(x) = w(x', x_n) = 2L - 2L|x'|$$

for $x = (x', x_n) \in (B_1(0)\backslash B_{1/2}(0)) \times \mathbb{R} \subset S$. Since w is superharmonic with $w = 0$ on $\partial B_1(0) \times \mathbb{R}$ and $w = L \geq \sup u_{0\epsilon}$ on $\partial B_{1/2}(0) \times \mathbb{R}$, a comparison yields that $w \geq u_R$ which implies (ii). Now (1.3) – the Lipschitz continuity of u_ϵ in x implies Hölder continuity in t with exponent $1/2$ (see [5, 11]). Since u_ϵ converges uniformly to u along a sequence, one obtains $u \in C^{1,1/2}$.

For the proof of Theorem 1.1, we will construct sufficiently small parabolic cubes near the contact point so that the free boundary of u becomes "flat" and "does not intersect" ∂S, more precisely, u becomes a solution of (P) in those small cubes with flat boundaries. Then by [1], the limit solution u is also a domain variation solution, and hence we can apply the regularity results in [1]. Also in the proof, we will use comparison principle for viscosity solutions, which was proved in [8]. We refer to Theorem 1.3 of [8] for the proof that every limit solution is a viscosity solution.

2. Preliminary Lemmas

Below we state some properties of caloric and harmonic functions defined in Lipschitz domains, a comparison principle for (P), existence of a radially symmetric solution of (P), and a regularity result for solutions with flat boundaries.

Lemma 2.1 [3, Lemma 5]. *Let Ω be a Lipschitz domain in $\mathbb{R}^n \times \mathbb{R}$ such that $0 \in \partial\Omega$, i.e.,*

$$Q_1(0) \cap \Omega = Q_1(0) \cap \{(x, t) : x_n < f(x', t)\},$$

where we denote $x = (x', x_n)$ for $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$, f satisfies $|f(x, t) - f(y, s)| \leq L(|x - y| + |t - s|)$ for some $L > 0$, and $f(0, 0) = 0$. Let u be a positive caloric function in $Q_1(0) \cap \Omega$ such that $u = 0$ on $\partial\Omega$, $u(-e_n, 0) = m_1 > 0$ and $\sup_{Q_1(0)} u = m_2$. Then there exist $a > 0$ and $\delta > 0$ depending only on n, L, $\frac{m_1}{m_2}$ such that

$$w_+ := u + u^{1+a} \quad \textit{and} \quad w_- := u - u^{1+a}$$

are subharmonic and superharmonic, respectively, in $Q_\delta \cap \Omega \cap \{t = 0\}$.

Lemma 2.2 [7]. *Let u_1 and u_2 be two nonnegative harmonic functions in a domain D of $\mathbb{R}^n$ of the form*

$$D = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x'| < 2, |x_n| < 2L, x_n < f(x')\}$$

where f is a Lipschitz function with constant L and $f(0) = 0$. Assume further that $u_1 = u_2 = 0$ along the graph of f, and

$$\frac{u_1(0, L)}{u_2(0, L)} = 1.$$

Then, $u_1(x', x_n)/u_2(x', x_n)$ is Hölder continuous in $\overline{D_{1/2}}$ for some coefficient α, both α and the C^α norm of u_1/u_2 depending only on L. Here $D_{1/2}$ denotes $\{|x'| < 1, |x_n| < L, x_n < f(x')\}$.

Lemma 2.3 [5, Proposition 1.1]. *There exists a radially symmetric solution $U(x, t)$ of* (P) *in the form*

$$U(x, t) = (T_0 - t)^{1/2} f(|x|(T_0 - t)^{-1/2})$$

where f is a concave C^1-function on $[0, K]$ *with $f(K) = 0$ and $f'(K) = -1$.*

Theorem 2.4 [8, Theorems 1.3 and 2.2]. *Let u and v be, respectively, a sub- and supersolutions of* (P) *with strictly separated initial data $u_0 \prec v_0$. Then the solution remain ordered for all time*

$$u(x, t) \prec v(x, t) \quad \text{for every } t > 0.$$

The following theorem is a regularity result obtained for a domain variation solution. (See [11] or [1] for the definition of a domain variation solution.)

Theorem 2.5 [1, Theorem 8.4]. *Let (u, χ) be a domain variation solution of* (P) *in $Q_\rho := Q_\rho(0, 0)$ such that $(0, 0) \in \partial\{u > 0\}$. There exists a constant $\sigma_1 > 0$ such that if $u(x, t) = \chi(x, t) = 0$ when $(x, t) \in Q_\rho^-$ and $x_n \geq \sigma\rho$, and if $|\nabla u| \leq 1 + \tau$ in Q_ρ^- for some $\sigma \leq \sigma_1$ and $\tau \leq \sigma_1 \sigma^2$, then the free boundary $\partial\{u > 0\}$ is in $Q_{\rho/4}^-$ the graph of a $C^{1+\alpha,\alpha}$-function*; *in particular the space normal is Hölder continuous in $Q_{\rho/4}^-$.*

3. Proof of Theorem

Let $Q_{r_0} = B_{r_0}(z) \times (0, r_0^2) \subset \mathbb{R}^{n+1}$ with $z \in \partial S$, and let u solve (PS) in Q_{r_0}. Suppose $u \in C^{1,1/2}(Q_{r_0})$ with a norm $A > 0$.

Proof of (A). Denote by $\frac{1}{2}S$, the middle cylinder in S, that is

$$\frac{1}{2}S = B_{1/2}(0) \times \mathbb{R} \subset S.$$

Let h be the harmonic function in $S - \frac{1}{2}S$ such that

$$\begin{cases} h = 0 & \text{on } \partial S \\ h = c_n & \text{on } \partial \frac{1}{2} S \\ \frac{\partial h}{\partial \nu} = 1 & \text{on } \partial S \end{cases} \tag{3.1}$$

where $c_n > 0$ is a dimensional constant and ν is inward unit normal to ∂S. Define a set $\mathcal{B}$ as below

$$\mathcal{B} = \partial S \times (0, \infty) \backslash \mathcal{C}$$

where $(x, t) \in \mathcal{C} \subset \partial S \times (0, \infty)$ if and only if $u \leq Ch$ in $(B_r(x) \cap S) \times (\tau, t)$ for some $r > 0$, $\tau < t$ and $C < 1$. Throughout the proof, we fix a contact point $(x_0, T) \in Q_{r_0}$ such that

$$x_0 \in \Gamma_T(u) \cap \partial S \tag{3.2}$$

and will prove the theorem near the point (x_0, T).

First observe that $(x_0, T) \in \mathcal{B}$. (Otherwise, $(x_0, T) \in \mathcal{C}$ and one can construct a supersolution $w \geq u$ such that $w = 0$ in $B_c(x_0) \times (\tau + (t - \tau)/2, t)$ for sufficiently small $c > 0$ depending on C. This would contradict (3.2).)

Let v be a caloric function in $S \cap B_{r_0}(z) \times (0, r_0^2)$ such that

$$\begin{cases} v_t - \Delta v = 0 & \text{in } (S \cap B_{r_0}(z)) \times (0, r_0^2) \\ v = 0 & \text{on } (\partial S \cap B_{r_0}(z)) \times (0, r_0^2) \\ v = u & \text{on } \partial B_{r_0}(z) \times (0, r_0^2) \cup \{t = 0\}. \end{cases}$$

Then by Lemma 2.1, there exist $a = a(n, A) > 0$ and $\delta = \delta(n, A) > 0$ such that $(v + v^{1+a})(\cdot, t)$ is subharmonic in $B_{\delta\sqrt{T}}(x_0)$ for any $t \in ((1 - \delta^2)T, T)$. (Here recall that the contact point $(x_0, T) \in Q_{r_0} = B_{r_0}(z) \times (0, r_0^2)$.) Then for $x \in B_{\delta\sqrt{T}}(x_0)$ and $t \in ((1 - \delta^2)T, T)$

$$u(x, t) \leq v(x, t) \leq C_1 h(x) \tag{3.3}$$

where the first inequality follows from the comparison, the second inequality follows from the subharmonicity of $v + v^{1+a}$ and Lemma 2.2 for some $C_1 > 0$. Since $(x_0, T) \in \mathcal{B}$, one can observe that (3.3) holds for

$$C_1 \geq 1.$$

Now suppose that there is an angle $0 < \theta < \pi/2$ and a positive sequence $\{r_i\}$ converging to 0 such that $r_{i+1} \leq r_i/20$ and

$$\Lambda(x_0, \theta) \cap \partial B_{r_i}(x_0) \not\subseteq \Omega_T, \quad \forall i \in \mathbb{N}.$$

Construct time intervals $0 < t_1 < t_2 < \cdots < t_m < t_{m+1} < T$ such that

$$t_i = T - \frac{1}{10}\sum_{j=i}^{m+1} r_j^2$$

where m is an integer, which will be chosen later. Then we get

$$T - \frac{r_i^2}{5} \leq t_i \leq T - \frac{r_i^2}{10}.$$

Also we may assume r_1 is sufficiently small so that

$$2r_1 \leq \delta\sqrt{T}, \quad T - \delta^2 T \leq T - \frac{r_1^2}{5} \leq t_1. \tag{3.4}$$

We will construct a sequence of functions $w_i = C_i h$ in $B_{2r_{i+1}}(x_0) \times [t_{i+1}, T]$ for $0 \leq i \leq m$ such that $w_i \geq u$ in $B_{2r_{i+1}}(x_0) \times [t_{i+1}, T]$ and $\partial w_m/\partial \nu < 1$ on $\partial S \times [t_{m+1}, T]$. Then it would yield that (x_0, T) is not in $\mathscr{B}$.

First, let $w_0 = C_1 h$ then by (3.3) and (3.4),

$$w_0 = C_1 h \geq u \tag{3.5}$$

in $B_{2r_1}(x_0) \times [t_1, T]$. To construct w_i for $i \geq 1$, let

$$z_i \in \Lambda(x_0, \theta) \cap \partial B_{r_i}(x_0) \cap \{u(\cdot, T) = 0\}$$

and let $c = \frac{\cos\theta}{10A}$. Then we get

$$B_{cr_i}(z_i) \subset S, \quad \operatorname{dist}(B_{cr_i}(z_i), \partial S) \geq \frac{r_i \cos\theta}{2}. \tag{3.6}$$

Observe that in $B_{cr_i}(z_i) \times [t_i, T]$

$$u \leq \frac{r_i \cos\theta}{5} \tag{3.7}$$

since A is a $C^{1,1/2}$ norm of u and $u(z_i, T) = 0$. On the other hand, if $C_i \geq 1$, then the inequality in (3.6) implies that in $B_{cr_i}(z_i)$

$$C_i h \geq \frac{r_i \cos\theta}{2}. \tag{3.8}$$

Now let

$$g_1(x, t) = C_1 h(x) - u(x, t)$$

then g_1 solves the heat equation in $B_{2r_1}(x_0) - B_{cr_1}(z_1) \times [t_1, T]$, and (3.5) implies that

$$g_1 \geq 0 \quad \text{on } \{t = t_1\} \cup \partial B_{2r_1}(x_0).$$

Also (3.7) and (3.8) imply that on $\partial B_{cr_1}(z_1) \times [t_1, T]$

$$g_1 \geq \frac{3r_1 \cos\theta}{10} \geq C_\theta h \tag{3.9}$$

for a constant C_θ depending only on θ and n. Since $g_1 = 0$ on $\partial S \times [t_1, T]$, Lemmas 2.1, 2.2 and (3.9) yield that there exists a constant $\lambda = \lambda(\theta, A, n) > 0$ such that in $B_{r_1/10}(x_0) \times [T - r_1^2/100, T]$

$$g_1 \geq \lambda h. \tag{3.10}$$

Since $2r_2 \leq r_1/10$ and $t_2 \geq T - r_2^2/5 \geq T - r_1^2/100$, (3.10) implies that in $B_{2r_2}(x_0) \times [t_2, T]$

$$w_1 := (C_1 - \lambda)h \geq C_1 h - g_1 = u. \tag{3.11}$$

Let m be the integer satisfying $C_1 - m\lambda < 1 \leq C_1 - (m-1)\lambda$ and let $w_i = (C_1 - i\lambda)h$ for $1 \leq i \leq m$. Then by a similar reasoning as in the proof of (3.11), we obtain $u \leq w_i$ for $1 \leq i \leq m$ in $B_{2r_{i+1}}(x_0) \times [t_{i+1}, T]$. In particular,

$$u \leq w_m$$

in $B_{2r_{m+1}}(x_0) \times [t_{m+1}, T]$. Here we observe

$$\partial w_m/\partial \nu = (C_1 - m\lambda)\partial h/\partial \nu = C_1 - m\lambda < 1$$

in $B_{r_{m+1}}(x_0) \times [t_{m+1}, T]$, which contradicts $(x_0, T) \in \mathcal{B}$.

Proof of (B). Let $\sigma_n > 0$ be a sufficiently small dimensional constant, which will be chosen later. Let $x_0 \in \Gamma_T \cap \partial S$ and let $|\nabla u| \leq 1 + \sigma_n$ on $\partial S \cap Q_\rho^-(x_0, T)$ for some $\rho > 0$.

1. First, we will find an upper bound of $|\nabla u|$ inside a positive set of u near (x_0, T), and from which we will obtain $u \leq (1 + 4\sigma_n)h$ near (x_0, T). Let w solve the heat equation in $Q_\rho^-(x_0, T) \cap S$ with

$$w = \begin{cases} \max\{|\nabla u|^2 - 1, 0\} & \text{on } \{t = T - \rho^2\} \cup (\partial B_\rho(x_0) \cap S) \\ 0 & \text{on } \partial S. \end{cases}$$

(Here, we let $|\nabla u| = 0$ outside the positive set Ω of u.) Since $|\nabla u| \leq 1 + \sigma_n$ on $\partial S \cap Q_\rho^-(x_0, T)$, we get

$$\max\{|\nabla u|^2 - 1, 0\} \leq w + 3\sigma_n$$

on the parabolic boundary of $S \cap Q_\rho^-(x_0, T)$. Since $\max\{|\nabla u|^2 - 1, 0\}$ is a subcaloric function in $\Omega \cap Q_\rho^-(x_0, T)$ with boundary value 0 on Γ, it follows by comparison that

$$|\nabla u|^2 - 1 \leq w + 3\sigma_n \quad \text{in } \Omega \cap Q_\rho^-(x_0, T). \tag{3.12}$$

On the other hand, Lemma 2.1 implies that there exists $a = a(A, n) > 0$ such that $w + w^{1+a}$ is a positive subharmonic function in $Q^-_{\rho/2}(x_0, T)$. Since $w + w^{1+a}$ vanishes on ∂S, there exists a small $\eta = \eta(\rho, \sigma_n, A, n) > 0$ such that

$$w \leq \sigma_n \quad \text{in } Q^-_\eta(x_0, T). \tag{3.13}$$

Then (3.12) and (3.13) imply

$$\max_{Q^-_\eta(x_0,T)} |\nabla u| \leq 1 + 4\sigma_n \tag{3.14}$$

which yields that in $Q^-_\eta(x_0, T)$

$$u(x, t) \leq (1 + 4\sigma_n)\text{dist}(x, \partial S) \leq (1 + 4\sigma_n)h(x). \tag{3.15}$$

2. Let $0 < \theta_0 < \pi/2$ be a dimensional constant close to $\pi/2$, which will be chosen later. With a choice of a sufficiently small $\sigma_n = \sigma_n(\theta_0, A, n) > 0$, we will show that for any $(x, t) \in \mathcal{B} \cap Q^-_{\eta/4}(x_0, T)$, Ω_t contains a space cone with vertex x and opening $2\theta_0$, more precisely,

$$\Lambda(x, \theta_0) \cap B_{\eta/4}(x) \subset \Omega_t \quad \text{for } (x, t) \in \mathcal{B} \cap Q^-_{\eta/4}(x_0, T). \tag{3.16}$$

In other words, we will prove that (A) of Theorem 1.1 holds uniformly for points in $\mathcal{B} \cap Q^-_{\eta/4}(x_0, T)$.

For the proof of (3.16), let $\lambda = \lambda(\theta_0, A, n)$ be the constant given as in the proof of (A), which satisfies (3.10) and (3.11) for $\theta = \theta_0$. Let $\sigma_n > 0$ be sufficiently small so that $4\sigma_n < \lambda$. Suppose that (3.16) does not hold for some $(x, t) \in \mathcal{B} \cap Q^-_{\eta/4}(x_0, T)$. Then there exists a point $p \in \Lambda(x, \theta_0) \cap B_{\eta/4}(x) \cap \{u(\cdot, t) = 0\}$. Then by a similar proof as in (3.11) and by (3.15), we get

$$u \leq (1 + 4\sigma_n - \lambda)h \quad \text{in } Q^-_{|x-p|/10}(x, t). \tag{3.17}$$

Since $1 + 4\sigma_n - \lambda < 1$, it contradicts $(x, t) \in \mathcal{B}$.

3. Let $\epsilon = \epsilon(n) > 0$ be a sufficiently small dimensional constant. Let $0 < \sigma_0 = \sigma_1/2 < 1$ where σ_1 is the small constant given as in Theorem 2.5. Choose $\theta_0 = \theta_0(\epsilon)$ sufficiently close to $\pi/2$ so that

$$\cos\theta_0 \leq \epsilon^2. \tag{3.18}$$

By Steps 1 and 2, there exists η depending on ρ, σ_n, θ_0, A and n such that (3.16) holds in $Q^-_{\eta/4}(x_0, T)$. Now let

$$r = \epsilon\eta/2 \tag{3.19}$$

and let $y \in \Gamma_T(u) \cap B_r(x_0)$, $\text{dist}(y, \partial S) > 0$. We will show that $\Gamma_T(u)$ is a graph of $C^{1+\alpha,\alpha}$-function in $B_K(y)$ for some K much larger than $\text{dist}(y, \partial S)$. For simplicity, we will denote

$$\Gamma_t = \Gamma_t(u), \quad \Gamma = \Gamma(u) = \{(x, t) : x \in \Gamma_t\}.$$

Let $d = |y - x_0| \le r$, then by (3.16) and (3.18),

$$\sup\{\mathrm{dist}(z, \partial S) : z \in \Gamma_T \cap B_d(y)\} \le 2d\cos\theta_0 \\ \le 2\epsilon^2 d < \sigma_0 d. \tag{3.20}$$

if $\epsilon = \epsilon(\sigma_0)$ is sufficiently small. Since $\mathrm{dist}(y, \partial S) > 0$, we can take a positive constant $s < d$ such that

$$\sup\{\mathrm{dist}(z, \partial S) : z \in \Gamma_T \cap B_s(y)\} = \sigma_0 s. \tag{3.21}$$

We divide into the following two cases, then the regularity of Γ_T will follow from Theorem 2.5 in Case 1, and we show that Case 2 cannot happen.

Case 1. Suppose that Γ does not intersect $\mathscr{B}$ in $B_s(y) \times [T - s^2, T]$. Then u solves (P) in $Q_s^-(y, T)$ since every contact point of Γ is contained in $\mathscr{B}$. Denote $y = (y', y_n) \in B_1(0) \times \mathbb{R} = S$, and let ν be the inward unit normal to ∂S such that $\nu = -y'/|y'|$. Let P_ν be the hyperplane in $\mathbb{R}^n$ with a normal vector ν, and satisfying $\mathrm{dist}(P_\nu, \partial S) = 4\sigma_n s$. Precisely,

$$P_\nu = \{x : (x - \tilde{y}) \cdot \nu = 0\}$$

where $\cdot$ is the inner product and $\tilde{y} = y - (\mathrm{dist}(y, \partial S) + 4\sigma_n s)\nu$ is the projection of y to $\partial B_{1+4\sigma_n s}(0) \times \mathbb{R}$. Then for $(x, t) \in Q_s^-(y, T) \cap \Omega(u)$

$$u(x, t) \le (1 + 4\sigma_n)\mathrm{dist}(x, \partial S) \le \mathrm{dist}(x, P_\nu)$$

where the first inequality follows from (3.14) since $Q_s^-(y, T) \subset Q_\eta^-(x_0, T)$. Let $g(x, t) = \mathrm{dist}(x, P_\nu)$, then g solves (P) and its order parameter χ_g is the characteristic function $\chi_{\{(\tilde{y}-x)\cdot\nu<0\}}$. Since the order parameter χ of u is bounded above by χ_g, we obtain $u(x, t) = \chi(x, t) = 0$ when $(x, t) \in Q_s^-(y, T)$ and $(y - x)\cdot\nu \ge |y - \tilde{y}|$. We let $\sigma_n = \sigma_n(\theta_0, \sigma_0, A, n) > 0$ be sufficiently small so that $4\sigma_n \le \sigma_0^3$. Then

$$|y - \tilde{y}| = \mathrm{dist}(y, \partial S) + 4\sigma_n s \le \sigma_0 s + 4\sigma_n s < 2\sigma_0 s = \sigma_1 s$$

where the first inequality follows from (3.21). Also by (3.14),

$$|\nabla u| \le 1 + 4\sigma_n \le 1 + \sigma_0^3 < 1 + \sigma_1^3$$

in $Q_s^-(y, T)$. Hence by Theorem 2.5, we obtain that Γ_T is the graph of a $C^{1+\alpha,\alpha}$-function in $Q_{s/4}^-(y, T)$. Here observe that the radius $s/4$ is much larger than $\mathrm{dist}(y, \partial S)$ and σ_n depends on A and n since θ_0 and σ_0 are dimensional constants.

Case 2. Suppose that there exists a point

$$(z_0, t_0) \in \Gamma \cap \mathscr{B} \cap B_s(y) \times [T - s^2, T]. \tag{3.22}$$

Since $(z_0, t_0) \in \mathscr{B} \cap Q_{\eta/4}^-(x_0, T)$, (3.16) holds for (z_0, t_0) and a similar argument as in (3.20) yields that

$$\sup\{\mathrm{dist}(z, \partial S) : z \in \Gamma_{t_0} \cap B_{\sigma_0 s/\epsilon}(z_0)\} \le \frac{\sigma_0 s \cos\theta_0}{\epsilon} \le \epsilon\sigma_0 s. \tag{3.23}$$

On the other hand, the construction of s (see (3.21)) implies that there exists a point

$$\zeta \in \Gamma_T \cap B_s(y) \tag{3.24}$$

such that

$$\text{dist}(\zeta, \partial S) = \sigma_0 s.$$

Since $\zeta \in B_s(y) \subset B_{2s}(z_0)$, (3.23) implies that

$$\sup\{\text{dist}(z, \partial S) : z \in \Gamma_{t_0} \cap B_{\sigma_0 s/2\epsilon}(\zeta)\} \leq \epsilon\sigma_0 s. \tag{3.25}$$

Now observe that, in the large ball $B_{\sigma_0 s/2\epsilon}(\zeta)$, the free boundary Γ_{t_0} stays $\epsilon\sigma_0 s$-close to ∂S. In other words, Γ_{t_0} is ϵ^2-flat in a ball with radius comparable to s/ϵ. But since Γ_T contains ζ, and ζ is $\sigma_0 s$-away from ∂S, it implies that the free boundary should move by more than $\sigma_0 s/2$ on the time interval $[t_0, T]$, near the point ζ. Here we recall that $T - t_0$ is sufficiently small, i.e., $T - t_0 \leq s^2$. Then to derive a contradiction, we show that it takes more time than $2s^2$ for $\Gamma_t(u)$ to move by $\sigma_0 s/2$ near ζ. More precisely, using the ϵ^2-flatness of Γ_{t_0}, we construct a radially symmetric solution v in $\Omega_{t_0}(u) \times [t_0, T]$ such that $v \leq u$ and $v(\zeta, T) > 0$, which would contradict $\zeta \in \Gamma_T(u)$.

Let S' be the concentric subcylinder of S such that $\text{dist}(\partial S', \partial S) = \epsilon\sigma_0 s/2$, i.e.,

$$S' = B_{1-\epsilon\sigma_0 s/2}(0) \times \mathbb{R}.$$

First, we show that $u(\cdot, t_0)$ is bounded below by $(1-\epsilon)h(\cdot)$ in $S' \cap B_{\sigma_0 s/\epsilon}(\zeta)$. Since $Q^-_{\sigma_0 s/\epsilon}(\zeta, t_0) \subset Q_\eta(x_0, T)$, (3.15) implies

$$u(x, t) \leq (1 + 4\sigma_n)h(x)$$

in $Q^-_{\sigma_0 s/\epsilon}(\zeta, t_0)$. Suppose that for some $y \in S' \cap B_{\sigma_0 s/\epsilon}(\zeta)$,

$$u(y, t_0) \leq (1-\epsilon)h(y).$$

Then by a similar reasoning as in (3.17), there exists $\lambda = \lambda(\epsilon, \sigma_0, n) > 0$ such that

$$u \leq (1 + 4\sigma_n - \lambda)h \tag{3.26}$$

in $Q^-_{\sigma_0 s/4\epsilon}(\zeta, t_0) \cap \{x : \text{dist}(x, \partial S) \leq \epsilon\sigma_0 s/10\}$. Here note that λ is a dimensional constant as well as ϵ and σ_0. If we choose a sufficiently small σ_n such that $4\sigma_n < \lambda$, then (3.26) would yield that $\partial S \cap B_{\sigma_0 s/8\epsilon}(\zeta)$ does not intersect $\mathcal{B} \cap \{t = t_0\}$. This would contradict $(z_0, t_0) \in \mathcal{B}$ since $z_0 \in \partial S \cap B_{2s}(\zeta)$. Hence we conclude

$$u(\cdot, t_0) \geq (1-\epsilon)h(\cdot) \quad \text{in } S' \cap B_{\sigma_0 s/\epsilon}(\zeta). \tag{3.27}$$

Now we construct a radially symmetric solution v on the time interval $[t_0, T]$ such that $v \leq u$ on $[t_0, T]$, and $v(\zeta, T) > 0$. Denote $\zeta = (\zeta', \zeta_n) \in B_1(0) \times \mathbb{R}$ and let v

be the inward unit normal vector to ∂S such that $\nu = -\zeta'/|\zeta'|$. Let

$$R = \frac{\sigma_0 s}{4\epsilon}, \quad \zeta_\epsilon = \zeta - \left(\sigma_0 s - \frac{\epsilon\sigma_0 s}{2}\right)\nu \in \partial S', \quad \zeta_R = \zeta_\epsilon + R\nu$$

where ζ_ϵ is the projection of ζ to $\partial S'$. Let Σ be the ball with center at ζ_R and with radius R, i.e.,

$$\Sigma = B_R(\zeta_R)$$

then we get

$$\Sigma \subset S' \cap B_{\sigma_0 s/2\epsilon}(\zeta), \quad \zeta \in \Sigma, \quad \mathrm{dist}(\zeta, \partial\Sigma) = \left(1 - \frac{\epsilon}{2}\right)\sigma_0 s.$$

In particular, (3.27) implies

$$u(\cdot, t_0) \geq (1-\epsilon)h(\cdot) \quad \text{in } \Sigma.$$

Let $\tilde{v}$ be the radially symmetric solution of (P) with center at ζ_R. Then by Lemma 2.3,

$$\tilde{v}(x,t) = (T_0 - t)^{1/2} f((T_0 - t)^{-1/2}|x - \zeta_R|)$$

where $T_0 > 0$ is a constant, and f is a decreasing concave function on $[0, K)$ with $f(K) = 0$ and $f'(K) = -1$. Let $\tau_0 \in (0, T_0)$ be the time satisfying

$$(T_0 - \tau_0)^{-1/2} R = K \tag{3.28}$$

then we get

$$\{\tilde{v}(\cdot, \tau_0) > 0\} = \Sigma.$$

Now let

$$v(x,t) = (1-\epsilon)\tilde{v}\left(\frac{x - \zeta_R}{1-\epsilon} + \zeta_R, \frac{t - t_0}{(1-\epsilon)^2} + \tau_0\right).$$

Then v solves

$$\begin{cases} v_t - \Delta v = 0 & \text{in } \{v > 0\} \\ |\nabla v| = 1 & \text{on } \partial\{v > 0\} \end{cases}$$

for $t \geq t_0$, and at the time $t = t_0$,

$$\begin{aligned} \{v(\cdot, t_0) > 0\} &\subset \Sigma \\ &\subset S' \cap B_{\sigma_0 s/2\epsilon}(\zeta) \\ &\subset \{u(\cdot, t_0) > 0\} \end{aligned}$$

where the last follows from (3.27). Also recall that $u(\cdot, t_0) \geq (1-\epsilon)h(\cdot)$ in Σ, where $h(x) = h(x', x_n) = h(|x'|)$ is a decreasing convex function on $[1/2, 1]$ with $h(1) = 0$ and $|h'(1)| = 1$. Since $\tilde{v}$ is radially symmetric with center at ζ_R, and it is concave with $|\nabla\tilde{v}(\cdot, \tau_0)| = 1$ on $\partial\Sigma$, we obtain that in Σ

$$\begin{aligned} u(\cdot, t_0) &\geq (1-\epsilon)h(\cdot) \geq (1-\epsilon)\tilde{v}(\cdot, \tau_0) \\ &\geq (1-\epsilon)\tilde{v}\left(\frac{\cdot - \zeta_R}{1-\epsilon} + \zeta_R, \tau_0\right) = v(\cdot, t_0). \end{aligned} \tag{3.29}$$

Hence by comparison, $u \geq v$ for $t \geq t_0$.

On the other hand, observe that $\zeta \in \{v(\cdot, t_0) > 0\}$ and

$$\begin{aligned} \mathrm{dist}(\zeta, \Gamma_{t_0}(v)) &= \mathrm{dist}(\zeta, \partial S) - \mathrm{dist}(\partial S, \Gamma_{\tau_0}(\tilde{v})) - \mathrm{dist}(\Gamma_{\tau_0}(\tilde{v}), \Gamma_{t_0}(v)) \\ &= \sigma_0 s - \frac{\epsilon\sigma_0 s}{2} - \frac{\sigma_0 s}{4} = \frac{(3-2\epsilon)\sigma_0 s}{4}. \end{aligned}$$

Let $t_0 + \alpha$ be the time the free boundary of v reaches the point ζ, i.e., let

$$\zeta \in \Gamma_{t_0+\alpha}(v).$$

Then since $\mathrm{dist}(\zeta, \Gamma_{t_0}(v)) = \frac{(3-2\epsilon)\sigma_0 s}{4} > \frac{\sigma_0 s}{2(1-\epsilon)}$, the construction of v yields that

$$\frac{R}{(T_0-\tau_0)^{1/2}} < \frac{R + (\sigma_0 s/2)}{(T_0 - (\tau_0 + \alpha(1-\epsilon)^{-2}))^{1/2}}. \tag{3.30}$$

Hence

$$\begin{aligned} \alpha &> (1-\epsilon)^2(T_0-\tau_0)\left(\frac{\sigma_0 s}{R} - \frac{\sigma_0^2 s^2}{4R^2}\right) \\ &> (1-\epsilon)^2(T_0-\tau_0)\frac{\sigma_0 s}{2R} \\ &= (1-\epsilon)^2\frac{\sigma_0 s R}{2K^2} = (1-\epsilon)^2\frac{\sigma_0^2 s^2}{8\epsilon K^2} > 2s^2 \end{aligned} \tag{3.31}$$

where the first inequality follows from (3.30), the first equality follows from (3.28), the second equality follows from $R = \sigma_0 s/4\epsilon$ and the last inequality follows if $\epsilon = \epsilon(\sigma_0, K) > 0$ is a sufficiently small dimensional constant. Since v is strictly decreasing in time, (3.31) implies

$$v(\zeta, t_0 + 2s^2) > 0.$$

Recall that by (3.24), $\zeta \in \Gamma_T(u)$, and by (3.22), $T \leq t_0 + s^2$. Since v is decreasing in time,

$$0 < v(\zeta, t_0 + 2s^2) \leq v(\zeta, T) \leq u(\zeta, T) = 0.$$

Hence we conclude that Case 2 cannot happen.

Proof of (C). Suppose that u is decreasing in time. Let $(x_0, T) \in \mathscr{B}$ and let $\theta_0 > 0$ be a dimensional constant, which will be chosen later. By (A), there exists $r > 0$ such that the cone $\Lambda(x_0, \theta_0) \cap B_r(x_0)$ is contained in Ω_T. Since u is decreasing in time, for $t \leq T$

$$\Lambda(x_0, \theta_0) \cap B_r(x_0) \subset \Omega_t$$

and hence

$$S' \cap B_r(x_0) \subset \Omega_t \tag{3.32}$$

where

$$S' = B_{1-r\cos\theta_0}(0) \times \mathbb{R} \subset S.$$

Let w_1 and w_2 be caloric functions in $S' \cap B_r(x_0) \times [T - r^2, T]$ such that

$$w_1 = \begin{cases} 0 & \text{on } \partial S' \\ u & \text{on } \partial B_r(x_0) \cup \{t = T - r^2\} \end{cases}$$

and

$$w_2 = \begin{cases} 2C_1 r\cos\theta_0 & \text{on } \partial S' \\ u & \text{on } \partial B_r(x_0) \cup \{t = T - r^2\} \end{cases}$$

where C_1 is the constant given as in (3.3). Then by (3.3) and (3.32),

$$w_1 \leq u \leq w_2 \leq w_1 + 2C_1 r\cos\theta_0 \tag{3.33}$$

in $S' \cap B_r(x_0) \times [T - r^2, T]$.

On the other hand, by Lemma 2.1, there exists $a = a(n, A) > 0$ such that $w_1 + w_1^{1+a}$ is subharmonic and $w_1 - w_1^{1+a}$ is superharmonic in $S' \cap B_{r/2}(x_0) \times [T - (r/2)^2, T]$. For $t \in [T - (r/2)^2, T]$, let g_t be the harmonic function in $S' \cap B_{r/2}(x_0)$ with

$$g_t = \begin{cases} 0 & \text{on } \partial S' \\ w_1(\cdot, t) & \text{on } \partial B_{r/2}(x_0). \end{cases}$$

Let

$$\epsilon = \sigma_n/10 > 0$$

then the almost harmonicity of w_1 implies that in $S' \cap B_{r/2}(x_0)$

$$(1 - \epsilon)g_t(\cdot) \leq w_1(\cdot, t) \leq (1 + \epsilon)g_t(\cdot)$$

if $r = r(\epsilon, A, n) > 0$ is chosen sufficiently small. Since g_t vanishes on $\partial S'$, Lemma 2.2 yields that there exists $0 < \delta = \delta(\epsilon, n) < 1$ such that in $S' \cap B_{\delta r}(x_0)$,

$$h_t \leq g_t \leq (1 + \epsilon)h_t$$

for a harmonic function h_t defined in $S' - \frac{1}{2}S$ with boundary values 0 on $\partial S'$, and a constant c_t on $\partial \frac{1}{2}S$. Here we let $\theta_0 = \theta_0(\epsilon, \delta, n)$ be sufficiently close to $\pi/2$ so that

$$\mathrm{dist}(x_0, \partial S') = r\cos\theta_0 \leq \epsilon\delta^2 r/10. \tag{3.34}$$

Combining the above inequalities, we get

$$(1-\epsilon)h_t(\cdot) \leq w_1(\cdot, t) \leq (1+2\epsilon)h_t(\cdot) \tag{3.35}$$

in $S' \cap B_{\delta r}(x_0)$. Then by (3.33) and (3.35),

$$(1-\epsilon)h_t(\cdot) \leq u(\cdot, t) \leq w_2(\cdot, t) \leq (1+2\epsilon)h_t(\cdot) + 2C_1 r\cos\theta_0 \tag{3.36}$$

in $S' \cap B_{\delta r}(x_0)$ and for $t \in [T - (r/2)^2, T]$.

Claim 1. *If $|\nabla h_{t_0}| \geq 1 + 5\epsilon$ on $\partial S'$ for some $t_0 \in [T - (r/2)^2, T]$, then*

$$\Omega_{t_0}(u) \cap B_{\delta r}(x_0) = S \cap B_{\delta r}(x_0).$$

For the proof of Claim 1, suppose $|\nabla h_{t_0}| \geq 1 + 5\epsilon$ on $\partial S'$ for $t_0 \in [T - (r/2)^2, T]$. Since w_1 is decreasing in time, (3.35) implies that in $B_{\delta r}(x_0) \times [T - r^2, t_0]$,

$$(1-\epsilon)h_{t_0}(\cdot) \leq w_1(\cdot, t). \tag{3.37}$$

Now we construct a subsolution v such that $v \leq u$ and $v(\cdot, t_0) > 0$ in $S \cap B_{\delta r}(x_0)$. Then it would yield that $S \cap B_{\delta r}(x_0) \subset \Omega_{t_0}(u)$. Let Ω' be the subset of $B_{\delta r}(x_0) \times [T - r^2, t_0]$ such that

$$\Omega' \cap \{t = T - r^2\} = S' \cap B_{\delta r}(x_0), \quad \Omega' \cap \{t = t_0\} = S \cap B_{\delta r}(x_0)$$

and that $\partial\Omega'$ has a constant normal velocity. (Here $S' \cap B_{\delta r}(x_0)$ is not empty since (3.34) implies $r\cos\theta_0 \ll \delta r$.)

Let v solve

$$\begin{cases} v_t - \Delta v = 0 & \text{in } \Omega' \\ v = (1-\epsilon)h_{t_0} & \text{on } \partial B_{\delta r}(x_0) \cup \{t = T - r^2\} \\ v = 0 & \text{on } \partial\Omega' \backslash \{t = T - r^2\}. \end{cases}$$

Then by (3.33) and (3.37), $v \leq u$ on $\partial B_{\delta r}(x_0) \cup \{t = T - r^2\}$, and $\Omega_{t_0}(v) \cap B_{\delta r}(x_0) = \Omega' \cap \{t = t_0\} = S \cap B_{\delta r}(x_0)$. Hence it suffices to show that v is a subsolution, i.e., $|\nabla v| \geq 1$ on $\partial\{v > 0\}$. Since Ω' is Lipschitz in parabolic scaling, Lemmas 2.1 and 2.2 imply that in $B_{\delta^2 r}(x_0) \cap S$

$$\alpha h(\cdot) \leq v(\cdot, t_0) \leq (1+\epsilon)\alpha h(\cdot) \tag{3.38}$$

for a constant $\alpha > 0$ and for the harmonic function h defined in (3.1), which vanishes on ∂S and $|\nabla h| = 1$ on ∂S. To find a lower bound of α, let $k(x, t) = (1-\epsilon)h_{t_0}(x)$.

Then k is caloric in $S' \cap B_{\delta r}(x_0) \times [T - r^2, t_0]$ and by comparison, $k \le v$ on $[T - r^2, t_0]$. Let y be a point in the middle of $S' \cap B_{\delta^2 r}(x_0)$ such that

$$\mathrm{dist}(y, \partial S') = \mathrm{dist}(y, \partial B_{\delta^2 r}(x_0)) \sim \delta^2 r.$$

Then at (y, t_0),

$$(1 + 4\epsilon)\mathrm{dist}(y, \partial S') \le (1 - \epsilon)h_{t_0}(y) = k(y, t_0) \le v(y, t_0)$$

where the first inequality follows from the convexity of h_{t_0} and $(1 - \epsilon)|\nabla h_{t_0}| \ge 1 + 4\epsilon$ on $\partial S'$. Then by (3.34) and (3.38)

$$(1 + 3\epsilon)\mathrm{dist}(y, \partial S) \le (1 + 4\epsilon)\mathrm{dist}(y, \partial S') \le v(y, t_0) \le (1 + \epsilon)\alpha h(y).$$

If $\delta^2 r$ is sufficiently small, then h is well-approximated by a linear function in $S \cap B_{\delta^2 r}(x_0)$ and by computation,

$$(1 + 3\epsilon)\mathrm{dist}(y, \partial S) \le (1 + \epsilon)\alpha h(y) \le (1 + 2\epsilon)\alpha\mathrm{dist}(y, \partial S)$$

which implies $1 + \epsilon \le \alpha$. Then by (3.38), $(1 + \epsilon)h \le v(\cdot, t_0)$ and we obtain

$$1 + \epsilon \le |\nabla v(\cdot, t_0)| \quad \text{on } \partial S \cap B_{\delta^2 r}(x_0).$$

By similar arguments, we also get $|\nabla v| > 1$ on $\partial\{v > 0\}$. Hence v is a subsolution and we can conclude the proof of Claim 1.

Claim 2. *For $x \in S' \cap B_{\delta r}(x_0)$ and $t \in [T - (r/2)^2, T]$*

$$u(x, t) \le w_2(x, t) \le (1 + 2\epsilon)h_t(x) + 2C_1 r \cos\theta_0$$

and for some $\eta > 0$

$$|\nabla h_t| \le 1 + 5\epsilon \quad \text{on } \partial S' \text{ for } t \in [T - \eta, T].$$

Since $x_0 \in \Gamma_T \cap \partial S$, there exists $0 < \eta < r/2$ such that $B_{\delta r}(x_0) \cap S \cap \Gamma_t(u) \ne \emptyset$ for $t \in [T - \eta, T]$. Then by Claim 1, $|\nabla h_t| \le 1 + 5\epsilon$ on $\partial S'$. The first part of Claim 2 follows from (3.36).

Claim 3. *For $x \in \partial S \cap B_{\delta r}(x_0)$ and $t \in [T - \eta, T]$,*

$$|\nabla u(x, t)| \le 1 + 10\epsilon = 1 + \sigma_n.$$

For the proof of Claim 3, we will construct a caloric function w_3 such that $u \le w_3$ in $S \cap B_r(x_0) \times [T - r^2, T]$, and using Claim 2, we will show that $|\nabla w_3| \le 1 + 10\epsilon$ on $\partial S \cap B_{\delta r}(x_0) \times [T - \eta, T]$. Let w_3 be a caloric function in $S \cap B_r(x_0) \times [T - r^2, T]$ such that

$$w_3 = \begin{cases} 0 & \text{on } \partial S \\ u & \text{on } \partial B_r(x_0) \cup \{t = T - r^2\}. \end{cases}$$

Then by comparison,

$$u \leq w_3 \tag{3.39}$$

in $S \cap B_r(x_0) \times [T - r^2, T]$. Also observe that for $x \in S' \cap B_{\delta r}(x_0)$ and $t \in [T - \eta, T]$,

$$w_3(x, t) \leq w_2(x, t) \leq (1 + 2\epsilon)h_t(x) + 2C_1 r \cos\theta_0 \tag{3.40}$$

where the first inequality follows from (3.3), and the second inequality follows from Claim 2.

On the other hand, by Lemmas 2.1 and 2.2, there exists $\alpha_t > 0$ depending on t such that for $x \in S \cap B_{\delta r}(x_0)$ and $t \in [T - (r/2)^2, T]$,

$$\alpha_t h(x) \leq w_3(x, t) \leq (1 + \epsilon)\alpha_t h(x). \tag{3.41}$$

Hence if v is the inward unit normal to ∂S at x_0, and if $t \in [T - \eta, T]$,

$$\begin{aligned}
\alpha_t h\left(x_0 + \frac{\delta r}{2}v\right) &\leq w_3\left(x_0 + \frac{\delta r}{2}v, t\right) \\
&\leq (1 + 2\epsilon)h_t\left(x_0 + \frac{\delta r}{2}v\right) + 2C_1 r \cos\theta_0 \\
&\leq (1 + 2\epsilon)(1 + 5\epsilon)h\left(x_0 + \frac{\delta r}{2}v\right) + 2C_1 r \cos\theta_0 \\
&\leq (1 + 8\epsilon)h\left(x_0 + \frac{\delta r}{2}v\right)
\end{aligned}$$

where the first inequality follows from (3.41), the second inequality follows from (3.40), the third inequality follows from Claim 2 and the convexity of h, and the last inequality follows if θ_0 is chosen sufficiently close to $\pi/2$ so that $\cos\theta_0 \leq \epsilon\delta/10C_1$. Hence we get

$$\alpha_t \leq 1 + 8\epsilon$$

for $t \in [T - \eta, T]$, and by (3.39) and (3.41)

$$|\nabla u| \leq |\nabla w_3| \leq (1 + \epsilon)\alpha_t < 1 + 10\epsilon = 1 + \sigma_n$$

on $\partial S \cap B_{\delta r}(x_0) \times [T - \eta, T]$. Then we can conclude that (C) of Theorem 3.25 holds for $\rho = \min\{\delta r, \sqrt{\eta}\} > 0$.

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